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A WELL-POSED OPTIMAL SPECTRAL ELEMENT
APPROXIMATION FOR THE STOKES PROBLEM

Y. Maday

A. T. Patera

E. M. Ronquist

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

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Y.MADAY*, A.T. PATERA**, E.M. RONQUIST**

ABSTRACT: In this paper we propose a new method for the spectral element simulation of incompressible flow. This method constitutes in a well-posed optimal approximation of the steady Stokes problem with no spurious modes in the pressure. The resulting method is analyzed, and numerical results are presented for a model problem.

*   Université de Paris VAL de MARNE et
    Laboratoire d'Analyse Numérique de l' Université Pierre et Marie CURIE,
    4, Place JUSSIEU, 75252, PARIS CEDEX 05.

** Department of Mechanical Engineering
    Massachusetts Institute of Technology
    CAMBRIDGE USA.

1. Introduction.

Two active areas of research in spectral methods are the definition of well-posed approximations to the Navier-Stokes equations in which rigorous error bounds can be obtained, and the construction of spectral-type techniques which are applicable to problems in general domains. As regards the former, recent progress has established "staggered-mesh" formulations resulting in well-posed, solvable schemes that are optimal in the velocity [27], [4], [5], [22], [2]. As regard spectral techniques for simulation of flows in complex domains, several algorithms have been proposed, including multidomain collocation patching schemes [11], [23], [20], [21], [15] and spectral element variational techniques [25], [13], [26].

The first schemes which involved staggered meshes, resulted in well-posed problems (i.e., without any spurious mode in the pressure) for one-dimensional problems [27], [4], however the extension of the ideas to higher space dimensions [22] introduced spurious modes into the system [2]. Although a workable scheme can be achieved by filtering the pressure [22], it is clearly desirable, in particular in three space dimensions, to construct a method in which the problem is intrinsically well-posed. To this end, a collocation technique is proposed in [2], in which the velocity and pressure spaces are chosen so as to give a unique solution. For this last scheme, an error analysis has been performed, and spectral accuracy is proved.

The spectral element spatial discretization involves a variational projection operator applied to elemental tensor product Lagrangian interpolants through local Chebyshev [25], [13] or Gauss-Lobatto Legendre collocation points [26]. The technique is capable of handling general geometries with relative ease, due to the "automatic" patching inherent in the variational formulation. In this paper, we consider a synthesis of the staggered-mesh and spectral element concepts that represents an enhancement of both ideas. The spectral element discretization benefits by a significant improvement in the treatment of the pressure as compared to past methods, in which either spurious pressure modes are present [25], or in which the pressure is treated inaccurately [13]. The staggered mesh schemes are improved in that the error estimates obtained

for the variational spectral element discretization are better than those obtained previously for collocation.

In Section 2, we present our numerical method as applied to a Legendre spectral element-Fourier discretization of the steady Stokes problem. All results presented here extend directly to higher space dimension, as well as to the unsteady case, as will be discussed in future papers (see e.g. [19]). In Section 3 a theoretical analysis is performed, in which it is shown that no spurious modes appear in the pressure. Furthermore, optimal error estimates are obtained for both the velocity and pressure. Lastly, in Section 4, we present some numerical results. These are in accordance with the theoretical estimates. We provide also some details on the numerical implementation of the method that uses a new algorithm for solving spectral Stokes discretizations, the details of which will be presented in a future paper[18].

In what follows, for any integer m and any domain Δ in $\mathbb{R}$ or $\mathbb{R}^2$, we denote by $C^m(\Delta)$ the space of all functions that are continuous over Δ as well as all their derivatives up to the order m, and by $C_0^\infty(\Lambda)$ the space of all functions that are infinitely differentiable with compact support in Λ.

In order to precise the sense in which the equations we shall consider have to be understood, we introduce some functional spaces. We denote by $L^2(\Delta)$ the standard Lebesgue space provided with the norm $\| . \|_\Delta$ and the scalar product $(.,.)_\Delta$ (or $\| . \|$ and $(.,.)$ when no confusion can occur). and for any positive real number r, the usual Sobolev space $H^r(\Delta)$ provided with the norm $\| . \|_{r,\Delta}$ and semi-norm $| . |_{r,\Delta}$ (or $\| . \|_r$ and $| . |_r$ when no confusion can occur).

Finally, C will stand for various constants that may vary from one line to the other, and for any function f depending on one variable x, we denote by f_x the derivative of f with respect to x.

2. The Numerical Method.

2.1 The model problem.

Throughout the paper, Ω is the domain $\Lambda \times \Theta$, with Λ = $]-1,1[$ and Θ = $]0,2\pi[$. The generic point in Ω will be denoted by $\mathbf{x}$ = (x,y). We consider the Stokes problem in the domain Ω for the velocity $\mathbf{u}$ = (u,v) and the pressure p, with no-slip boundary conditions in the first direction and periodic boundary conditions in the second. The problem is : *Find* $(\mathbf{u},p)$ *such that*

$$(2.1) \qquad \begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega , \\ \text{div } \mathbf{u} = 0 \quad \text{in } \Omega , \end{cases}$$

with the following boundary condition

$$(2.2) \qquad \forall\, y \in \Theta , \quad \mathbf{u}(-1,y) = \mathbf{u}(1,y) = 0 ,$$

$$(2.3) \qquad \forall\, x \in \Lambda , \quad \mathbf{u}(x,0) = \mathbf{u}(x,2\pi) .$$

Here ν is the kinematic viscosity, and $\mathbf{f}$ = (f,g) represents the density of body forces.

As is well-known, the appropriate space for the pressure is $L_0^2(\Omega)$ defined as follows

$$L_0^2(\Omega) = \{ \varphi \in L^2(\Omega) , \int_\Omega \varphi(\mathbf{x})\, d\mathbf{x} = 0 \} .$$

We denote by $C_\#^\infty(\Omega)$ (resp. $C_{0,\#}^\infty(\Omega)$) the space of all functions that are infinitely differentiable and are 2π-periodic in the second direction as well as their derivatives (resp. that are infinitely differentiable with compact support in the first direction and 2π-periodic in the second one as well as their derivatives). In order to take the boundary conditions (2.2)(2.3) into account we define the spaces $H_{0,\#}^1(\Omega)$ as the closure of $C_{0,\#}^\infty(\Omega)$ into $H^1(\Omega)$, and $H_0^1(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ into $H^1(\Lambda)$. Let us define also, for any positive real number r, the space $H_\#^r(\Omega)$ as the closure of $C_\#^\infty(\Omega)$ into $H^r(\Omega)$.

In this framework, it has been proved in [4] that the problem (2.1)(2.2)(2.3) is well posed for any force in the dual space $[H_\#^{-1}(\Omega)]^2$ of $[H_{0,\#}^1(\Omega)]^2$, the norm of which is denoted by $\| \cdot \|_{-1}$. More precisely we have

__Theorem 2.1__ : *For any* $\mathbf{f}$ = (f,g) *in* $[H_\#^{-1}(\Omega)]^2$ *, problem* (2.1)(2.2)(2.3) *has a unique solution* $(\mathbf{u},p)$ *in* $[H_{0,\#}^1(\Omega)]^2 \times L_0^2(\Omega)$ *, and one has*

(2.4) $\| \mathbf{u} \|_1 + \| p \| \leqslant C \| \mathbf{f} \|_{-1}$.

For any $\mathbf{f} = (f,g)$ in $[H_\#^\sigma(\Omega)]^2$, $\sigma \geqslant 0$, the solution $(\mathbf{u},p)$ of the problem (2.1)(2.2)(2.3) verifies

(2.5) $\| \mathbf{u} \|_{\sigma+2} + \| p \|_{\sigma+1} \leqslant C \| \mathbf{f} \|_\sigma$.

Let us write the dependent variables in terms of Fourier series in the periodic direction (we denote by $\underline{i}$ the complex square root of -1)

(2.6) $\begin{cases} \mathbf{u}(x,y) = \sum_{m=-\infty}^{\infty} \hat{\mathbf{u}}^m(x) \exp(\underline{i}\,my) \ , \\ p(x,y) = \sum_{m=-\infty}^{\infty} \hat{p}^m(x) \exp(\underline{i}\,my) \ . \end{cases}$

As is well known, this procedure decouples all Fourier modes for the Stokes problem, resulting in the following set of equations for the nth mode

(2.7)$_1$ $-\nu(\hat{u}_{xx}^n - n^2\hat{u}^n) + \hat{p}_x^n = \hat{f}^n$,

(2.7)$_2$ $-\nu(\hat{v}_{xx}^n - n^2\hat{v}^n) + \underline{i}\,n\hat{p}^n = \hat{g}^n$,

(2.8) $\hat{u}_x^n + \underline{i}\,n\hat{v}^n = 0$,

and

$\int_\Lambda \hat{p}^0(x) \, dx = 0$,

with the following boundary condition on $\hat{\mathbf{u}}^n$

(2.9) $\hat{\mathbf{u}}^n(-1) = \hat{\mathbf{u}}^n(1) = 0$.

The following proposition is now straightforward.

Proposition 2.1: For any $\mathbf{f} = (f,g)$ in $[H_\#^{-1}(\Omega)]^2$, the pair $(\mathbf{u},p)$ in $[H_{0,\#}^1(\Omega)]^2 \times L_0^2(\Omega)$ is the solution of problem (2.1)(2.2)(2.3) if and only if its Fourier modes are solutions in $[H_0^1(\Lambda)]^2 \times L^2(\Lambda)$ for $n \neq 0$ (resp. $[H_0^1(\Lambda)]^2 \times L_0^2(\Lambda)$ for $n = 0$) of problems (2.7)(2.8) (2.9).

Our numerical technique will be based on variational forms equivalent to (2.1)(2.2)(2.3) and to (2.7)(2.8)(2.9). We first introduce the notations

$X = [H_{0,\#}^1(\Omega)]^2$, $M = L_0^2(\Omega)$,

and provide X with the standard semi-norm $|\,.\,|_1$ of $[H^1(\Omega)]^2$ equivalent to $\|\,.\,\|_1$ over X.

The variational formulations are given by: *Find* $(\mathbf{u},p)$ *in* $X \times M$ *such that*

(2.10) $\nu(\nabla\mathbf{u},\nabla\mathbf{w}) - (p,\mathrm{div}\mathbf{w}) = (\mathbf{f},\mathbf{w})$, $\forall\,\mathbf{w}\in X$,

(2.11) $(q,\mathrm{div}\mathbf{u}) = 0$, $\forall\,q\in M$,

and (dropping the subscript n):*Find* $(\mathbf{u},p)$, $\mathbf{u}=(u,v)$ *in* $[H_0^1(\wedge)]^2 \times L^2(\Omega)$ *such that*

(2.12) $\nu[(u_x\,,w_x) + n^2\,(\mathbf{u},\mathbf{w})] - (p,w_x + \underline{i}\,nz) = (\mathbf{f},\mathbf{w})$, $\forall\,\mathbf{w}=(w,z)\in[H_0^1(\wedge)]^2$,

(2.13) $(q,u_x + \underline{i}\,nv) = 0$, $\forall\,q\in L^2(\wedge)$.

<u>Remark 2.1</u>: Note that, for the case n = 0, problem (2.12)(2.13) is well posed only in $[H_0^1(\wedge)]^2 \times L_0^2(\wedge)$; in what follows we shall not consider that case for simplicity of formulation.

2.2 The discrete formulation.

Let K be a fixed number independent of the forthcoming parameters of discretization. We divide $\wedge$ into K subintervals $\wedge_1,...,\wedge_K$, and set $\Omega_k = \wedge_k \times \Theta$. The spaces of approximation will consist of functions that are piecewise-polynomial over $\wedge_k$ and trigonometric in the second direction. These discrete functions will be determined in order to verify problem (2.10)(2.11) in a discrete sense. More precisely, we shall replace the integral appearing in the $L^2(\wedge)$-scalar product by quadrature formulas associated with the Gauss and Gauss-Lobatto points.

Let us introduce now the parameter of discretization h = (N,M), a pair of $\mathbb{N}^2$ with N $\geqslant$ 2. We denote by $\mathbb{P}_{N,K}$ the set of all functions that are polynomial of degree less than or equal to N on each subinterval $\wedge_k$, k = 1,...,K (in the case K $\square$ 1 we simply write $\mathbb{P}_N$).

Next, we denote by S_M , the set of all trigonometric polynomials of degree less than or equal to M , i.e.

$$S_M = \{\ \varphi(y) = \sum_{m=-M}^{M} \hat{\varphi}^m\,\exp(\underline{i}\,my)\ \})\ .$$

Let us define now the quadrature formulas on $\wedge$ and Θ. We denote by $(\zeta_i\,,\omega_i)$ for i = 1,..., N-1, the nodes and weights of the Gauss formula and by $(\xi_i\,,\varrho_i)$ for i = 0,..., N, the nodes

and weights of the Gauss–Lobatto formula. The following relations can be found in [8] for instance (L_n denotes the Legendre polynomial of degree n)

(2.14) $\quad \forall\, i \in \{1,...,N-1\}$, $L_{N-1}(\zeta_i) = 0$ and $L'_N(\xi_i) = 0$, $\xi_0 = -1$, $\xi_N = 1$,

$(2.15)_G$ $\quad \forall\, \varphi \in \mathbb{P}_{2N-3}$, $\sum_{i=1}^{N-1} \varphi(\zeta_i)\omega_i = \int_\Lambda \varphi(x)\,dx$,

$(2.15)_{GL}$ $\quad \forall\, \varphi \in \mathbb{P}_{2N-1}$, $\sum_{i=0}^{N} \varphi(\xi_i)\varrho_i = \int_\Lambda \varphi(x)\,dx$.

Over each subinterval Λ_k , $k = 1,...,K$, we then define a quadrature formula from the previous one's by a suitable affine transformation. Setting $\Lambda_k =]\,a_k\,,a_{k+1}\,[$, we define over Λ_k , the sets $(\zeta_{i,k}\,,\omega_{i,k})_{i=1,N-1}$ and $(\xi_{i,k}\,,\varrho_{i,k})_{i=0,N}$ as follows

(2.16)
$$\left\{ \begin{aligned} &\zeta_{i,k} = a_k + (a_{k+1} - a_k)(\zeta_i + 1)/2 \ , \\ &\omega_{i,k} = 2\omega_i/(a_{k+1} - a_k) \ , \\ &\xi_{i,k} = a_k + (a_{k+1} - a_k)(\xi_i + 1)/2 \ , \\ &\varrho_{i,k} = 2\varrho_i/(a_{k+1} - a_k) \ . \end{aligned} \right.$$

The previous points lead to the two discrete scalar products on $C^0(\overline\Lambda)$

$(2.17)_G$ $\quad \forall\, (\varphi,\psi) \in [C^0(\overline\Lambda)]^2$, $(\varphi\,,\psi)_{N,G} = \sum_{k=1}^{K}\sum_{i=1}^{N-1} \varphi(\zeta_{i,k})\psi(\zeta_{i,k})\omega_{i,k}$,

$(2.17)_{GL}$ $\quad \forall\, (\varphi,\psi) \in [C^0(\overline\Lambda)]^2$, $(\varphi\,,\psi)_{N,GL} = \sum_{k=1}^{K}\sum_{i=0}^{N} \varphi(\xi_{i,k})\psi(\xi_{i,k})\varrho_{i,k}$.

Over Θ, we consider the points $\theta_j = -\pi + 2j\pi/(2M+1)$, $j = 0,...,2M$. They verify

(2.18) $\quad \forall\, \varphi \in S_{2M}$, $(2\pi/(2M+1))\sum_{j=0}^{2M} \varphi(\theta_j) = \int_\Theta \varphi(y)\,dy$.

This gives us the following discrete scalar product over $C^0(\overline\Theta)$:

(2.19) $\quad \forall\, (\varphi,\psi) \in [C^0(\overline\Theta)]^2$, $(\varphi\,,\psi)_{M,\Theta} = (2\pi/(2M+1))\sum_{j=0}^{2M} \varphi(\theta_j)\psi(\theta_j)$.

Finally we define the discrete scalar products over $C^0(\overline\Omega)$:

$(2.20)_G$ $\quad \forall\, (\varphi,\psi) \in [C^0(\overline\Omega)]^2$, $(\varphi,\psi)_{h,G} = (2\pi/(2M+1))\sum_{k=1}^{K}\sum_{i=1}^{N-1}\sum_{j=0}^{2M} \varphi(\zeta_{i,k},\theta_j)\psi(\zeta_{i,k},\theta_j)\omega_{i,k}$

$(2.20)_{GL}$ $\quad \forall\, (\varphi,\psi) \in [C^0(\overline\Omega)]^2$, $(\varphi,\psi)_{h,GL} = (2\pi/(2M+1))\sum_{k=1}^{K}\sum_{i=0}^{N}\sum_{j=0}^{2M} \varphi(\xi_{i,k},\theta_j)\psi(\xi_{i,k},\theta_j)\varrho_{i,k}$

Let us set

$$X_h = X \cap (\mathbb{P}_{N,K} \otimes S_M)^2 \ , \quad M_h = M \cap (\mathbb{P}_{N-2,K} \otimes S_M) \ .$$

The discrete problem is : *Find* $(\mathbf{u}_h,p_h)$ *in* $X_h \times M_h$ *such that*

(2.21) $\quad \nu(\nabla\mathbf{u}_h,\nabla\mathbf{w})_{h,GL} - (p_h,\mathrm{div}\mathbf{w})_{h,G} = (\mathbf{f},\mathbf{w})_{h,GL}$, $\quad \forall\, \mathbf{w} \in X_h$,

(2.22) $\quad (q,\mathrm{div}\mathbf{u}_h)_{h,G} = 0$, $\quad \forall\, q \in M_h$.

Using the exactness of the quadrature formula (2.18) over $S_M \times S_M$ we can, as in the continuous case, decouple this problem into a set of $(2M + 1)$ equations for the Fourier modes of the solution: *Find* $(\mathbf{u}_N, p_N)$ *in* $[\mathbb{P}_{N,K} \cap H_0^1(\Lambda)]^2 \times \mathbb{P}_{N-2,K}$ *such that*

$$(2.23) \quad \nu[(\mathbf{u}_{Nx}, \mathbf{w}_x)_{N,GL} + n^2 (\mathbf{u}_N, \mathbf{w})_{N,GL}] - (p_N, w_x + i \, nz)_{N,G} = (\mathbf{f}, \mathbf{w})_{N,GL} \ ,$$

$$\forall \ \mathbf{w} = (w, z) \in [\mathbb{P}_{N,K} \cap H_0^1(\Lambda)]^2 \ ,$$

$$(2.24) \quad (q, u_{Nx} + i \, n \, v_N)_{N,G} = 0 \ , \quad \forall \, q \in \mathbb{P}_{N-2,K} \ .$$

<u>Remark 2.2</u> : We state here an equivalent pointwise interpretation of these problems. Let us first work with the case $K = 1$. We consider first a discrete problem close to (2.23) : *Find* u_N *in* $\mathbb{P}_N(\Lambda) \cap H_0^1(\Lambda)$ *such that*

$$(2.25) \quad (u_{Nx}, w_x)_{N,GL} + n^2 (u_N, w)_{N,GL} = (f, w)_{N,GL} \ , \quad \forall \ w \in \mathbb{P}_N \cap H_0^1(\Lambda) \ .$$

We note that the products $u_{Nx} w_x$ and $u_{Nxx} w$ belong to $\mathbb{P}_{2N-2}$ whence, from $(2.15)_{GL}$ and $(2.17)_{GL}$, we can write

$$(2.26) \quad (u_{Nx}, w_x)_{N,GL} = (u_{Nx}, w_x) = -(u_{Nxx}, w) = -(u_{Nxx}, w)_{N,GL} \ .$$

Let us now introduce the Lagrange interpolant Q_i of the point ξ_i , $i = 0,\ldots,N$, i.e. the polynomial of $\mathbb{P}_N$ that verifies

$$(2.27) \quad \forall \ i' = 0,\ldots,N \ , \quad Q_i(\xi_{i'}) = \delta_{ii'} \ ,$$

where $\delta_{ii'}$ denotes the Kronecker symbol. Taking $w = Q_i$, $i = 1,\ldots,N-1$, in (2.25) yields, thanks to (2.26),

$$(2.28) \quad \forall \ i = 1,\ldots,N-1 \ , \quad -u_{Nxx}(\xi_i) + n^2 u_N(\xi_i) = f(\xi_i) \ .$$

Problem (2.25) appears as a collocation approximation of the solution of the Poisson equation.

Unfortunately, the same is not true for the Stokes problem (2.23)(2.24). Indeed the discrete scalar product $(\, . \, , . \,)_{N,G}$ involves the points ζ_i . Hence, we should introduce, as test function z, the Lagrange interpolant corresponding to that set of points. Such an interpolant in the expression $[(\mathbf{u}_{Nx}, \mathbf{w}_x)_{N,GL} + n^2 (\mathbf{u}_N, \mathbf{w})_{N,GL}]$ would not decouple the Gauss-Lobatto points. The only equations we can obtain, in the case $K = 1$, are the following (take $\mathbf{w} = (Q_i, Q_j)$, $1 \leqslant i,j \leqslant N-1$, and note that $p_N Q_{ix}$ belongs to $\mathbb{P}_{2N-3}$ so that $(p_N, Q_{ix})_{N,G} = (p_N, Q_{ix}) = -(p_{Nx}, Q_i) = -(p_{Nx}, Q_i)_{N,GL}$)

$$\forall\, i = 1,...,N-1 \quad , \quad -\nu[u_{Nxx} - n^2 u_N](\xi_i) + p_{Nx}(\xi_i) = f(\xi_i) \quad ,$$

$$\forall\, i = 1,...,N-1 \quad , \quad -\nu[v_{Nxx} - n^2 v_N](\xi_i) + i\, \sum_{i'=0}^{N-1} Q_i(\zeta_{i'}) p_N(\zeta_{i'}) \omega_{i'} = f(\xi_i) \quad ,$$

$$\forall\, i = 1,...,N-1 \quad , \quad [u_{Nx} + i\, n v_N](\zeta_i) = 0 \quad .$$

Remark 2.3 : Let us now consider the case $K > 1$. Here, the interpretation of (2.25) involves the various virtual boundaries $a_{k+1} = \Lambda_k \cap \Lambda_{k+1}$, $k = 1,...,K-1$. Indeed taking the Lagrange interpolants of the points $\xi_{i,k}$ different of -1 and 1 gives

$$(2.29) \quad \forall\, k = 1,...,K \quad , \quad \forall\, i = 1,...,N-1 \quad , \quad -u_{Nxx}(\xi_{i,k}) + n^2\, u_N(\xi_{i,k}) = f(\xi_{i,k}) \quad ,$$

$$(2.30) \quad \forall\, k = 1,...,K-1 \quad , \quad [-u_{Nxx}(a_k) + n^2\, u_N(a_k) - f(a_k)](\varrho_{N,k-1} + \varrho_{0,k}) = u_{Nx}(a_k^-) - u_{Nx}(a_k^+) \quad ,$$

[here $g(x^+)$ (resp. $g(x^-)$) stands for $\displaystyle \lim_{\substack{t \to x \\ t > x}} g(t)$ (resp. $\displaystyle \lim_{\substack{t \to x \\ t < x}} g(t)$).]

Let us note that (2.29) is a collocation method for solving the Poisson equation, while (2.30) is, in a weak sense, the translation of $u_{Nx}(a_k^-) = u_{Nx}(a_k^+)$ since $\varrho_{N,k}$ and $\varrho_{0,k+1}$ are $\mathcal{O}(N^{-2})$. This condition on the derivatives means the continuity of u_{Nx} , which is the usual condition added to a multidomain technique.

For the problem (2.21)(2.22), we could derive a collocation-like interpretation of problem (2.12)(2.13) but this one is here not meaningfull on the boundary.

3. Error Analysis.

The main result of this section will consist in an asymptotic expression for the error bound between the exact solution of (2.10)(2.11) and the approximate solution of (2.21)(2.22). The result is optimal under the (mild) assumption that there exists a constant C^* independent of h such that

(3.1) $M \leqslant C^* N$.

This is not a limitation for the practical cases of numerical interest, but, for the theoretical point of view, we give in a final remark the behaviour of the error bound we can prove in the general case. We don't consider the dependence of the error bound with respect to a possible growth of K involving a decay of the measure of the Λ_k 's; this is now under consideration . The analysis is more technical since such a scheme would require a dependence of N and M with respect to the measure of the Λ_k 's and the various ratios of K, N and M would be involved in the estimate. Nethertheless if K is fixed or bounded, the case of dependence of N and M with respect to k can be handled by the same proofs as those explained in this section by using the general results of the appendix B.

The analysis we are going to perform will use extensively some of the main properties of the Legendre basis of polynomials. Let us recall them before starting the proofs. We denote by L_n the Legendre polynomial of degree n and recall that L_n has the same parity as n and that

(3.2) $L_n(-1) = (-1)^n$, $L_n(1) = 1$.

next we give the formulas that can be found in [8; Chapt.2,§7]

(3.3) $(L_n , L_{n'}) = (2/(2n+1)) \, \delta_{nn'}$,

(3.4) $((1-x^2)L_n')' + n(n+1)L_n = 0$,

(3.5) $(n+1) L_{n+1}(x) = (2n+1) x \, L_n(x) - nL_{n-1}(x)$,

(3.6) $\int_{-1}^{x} L_n(\zeta) \, d\zeta = [L_{n+1}(x) - L_{n-1}(x)]/(2n+1)$.

As pointed out in [5] the existence and uniqueness of the solution of problem (2.21)(2.22)

relay on a compatibility condition between the discrete spaces M_N and X_N. We first confirm this out in subsection 3.1 .

3.1 Some properties of the discrete divergence operator.

The main result of this subsection consists in the following

Lemma 3.1 : *For any* q *in* M_h *there exists a function* **w** *in* X_h *such that*

(3.7) $\forall k = 1,...,K$, $\forall i \square 1,...,N-1$, $\text{div } \mathbf{w}(\zeta_{i,k},.) = q(\zeta_{i,k},.)$,

(3.8) $\| \mathbf{w} \|_1 \leqslant C \| q \|$.

Proof of lemma 3.1: case $K = 1$. For any q in M_h , we define a polynomial Φ of $[\mathbb{P}_N \cap H_0^1(\wedge)] \otimes S_M$ by the following conditions

(3.9) $\forall i = 1,...,N-1$, $\Delta\Phi(\zeta_i,.) = q(\zeta_i,.)$.

We prove in the appendix A.1 that such a polynomial exists, is uniquely determined by (3.9) and satisfies the following bound

(3.10) $\| \Phi \|_2 \leqslant C \| q \|$.

Let us set $\tilde{\mathbf{w}} = (\tilde{w},\tilde{z}) = \text{grad } \Phi$, or again

(3.11) $\tilde{w} = \Phi_x$, $\tilde{z} = \Phi_y$.

As an easy consequence of (3.9)(3.10) we obtain

(3.12) $\begin{cases} \forall i \square 1,...,N-1 , \text{div } \tilde{\mathbf{w}}(\zeta_i,.) \square q(\zeta_i,.) , \\ \| \tilde{\mathbf{w}} \|_1 \leqslant C \| q \| . \end{cases}$

Let us search now a function Ψ in $(\mathbb{P}_N \otimes S_M) \oplus \mathbb{R}y$ such that

(3.13) $\Psi_y(\pm 1,.) \square - \Phi_x(\pm 1,.)$, $\Psi_x(\pm 1,.) = 0$.

To this purpose, let us write the functions Φ and Ψ in terms of Fourier series in the second direction, we derive

$$\Phi(x,y) = \sum_{m=-M}^{M} \hat{\Phi}^m(x) \exp(imy) , \hat{\Phi}^m \in \mathbb{P}_N \cap H_0^1(\wedge) ,$$

$$\Psi(x,y) = \sum_{m=-M}^{M} \hat{\Psi}^m(x) \exp(imy) + \lambda y , \hat{\Psi}^m \in \mathbb{P}_N \cap H_0^1(\wedge) .$$

As a consequence of the fact that q belongs to $L_0^2(\Omega)$, we obtain from $(2.15)_G$ and (3.9) ,

$$0 = \int_\Lambda dx \int_\Theta dy \; \Delta\Phi(x,y) = 2\pi \int_\Lambda dx \; (\hat{\Phi}^0_{xx})(x) = \hat{\Phi}^0_x(1) - \hat{\Phi}^0_x(-1) \; .$$

so that

$$(3.14) \quad \hat{\Phi}^0_x(1) = \hat{\Phi}^0_x(-1) \; .$$

Note also that due to the classical trace results (see [14]) we have

$$(3.15) \quad |\hat{\Phi}^0_x(1)| \leqslant C \, \| \Phi^0_x \|_{1/2,\Omega} \leqslant C' \, \| \Phi \|_{3/2,\Omega} \; .$$

Let us introduce, for any m in $\mathbb{N}$, two elements $r^\pm_m$ of $\mathbb{P}_N$ verifying

$$(3.16) \quad \begin{cases} r^+_m(1) = 1 \; , \; r^+_m(-1) = 0 \; , \; r^{+\prime}_m(\pm 1) = 0 \\ r^-_m(1) = 0 \; , \; r^-_m(-1) = 1 \; , \; r^{-\prime}_m(\pm 1) = 0 \; , \end{cases}$$

and such that there exists a constant C independent of m and N

$$(3.17) \quad (m^{-3} \| r^-_m{}'' \|^2 + m^{-1} \| r^-_m{}' \|^2 + m \| r^-_m \|^2) + (m^{-3} \| r^+_m{}'' \|^2 + m^{-1} \| r^+_m{}' \|^2 + m \| r^+_m \|^2) \leqslant C.$$

In order to find such elements, we define, for any μ in $\mathbb{N}$, the polynomials s_μ and $\tilde{s}_\mu$ by

$$s_\mu(x) = x^{2\mu} - (1/2) \, x^{4\mu} \; ,$$

$$\tilde{s}_\mu(x) = [(4\mu+1) \, x^{2\mu+1} - (2\mu+1) \, x^{4\mu+1}]/(4\mu) \; .$$

Then it is an easy matter to check that $r^\pm_m = s_{E(m/4C*)} \pm \tilde{s}_{E(m/4C*)}$ are elements of $\mathbb{P}_N$ and solutions of the problem (3.16)(3.17)(remind that the constant C* was introduced in (3.1) to impose a relation between M and N).

Let us define now Ψ by its components $\hat{\Psi}^m$ as follows

$$(3.18) \quad \begin{cases} \forall \, m \neq 0 \; , \; i m \, \hat{\Psi}^m = \hat{\Phi}^m_x(-1) \, r^-_m + \hat{\Phi}^m_x(1) \, r^+_m \; , \\ \hat{\Psi}^0 = 0 \; , \\ \lambda = -\hat{\Phi}^0_x(1) \; . \end{cases}$$

It is an easy matter to check now that, as a consequence of (3.14)(3.16), (3.13) is verified. Let us now estimate the H^2- norm of Ψ.

We begin by the H^2- seminorm of Ψ and compute it as follows

$$| \Psi |^2_2 = \| \Psi_{xx} \|^2 + \| \Psi_{xy} \|^2 + \| \Psi_{yy} \|^2$$

$$= \sum_{m=-M, m\neq 0}^{M} (1/m^2)(\| r^-_m{}'' \|^2 + m^2 \| r^-_m{}' \|^2 + m^4 \| r^-_m \|^2) \, |\hat{\Phi}^m_x(-1)|^2$$

$$+ \sum_{m=-M, m\neq 0}^{M} (1/m^2)(\| r^+_m{}'' \|^2 + m^2 \| r^+_m{}' \|^2 + m^4 \| r^+_m \|^2) \, |\hat{\Phi}^m_x(1)|^2$$

$$= \sum_{m=-M, m\neq0}^{M} (m^{-3} \| r_m^{-}{}'' \|^2 + m^{-1} \| r_m^{-}{}' \|^2 + m \| r_m^{-} \|^2) \, m \, | \hat{\Phi}_x^m(-1)|^2$$
$$+ \sum_{m=-M, m\neq0}^{M} (m^{-3} \| r_m^{+}{}'' \|^2 + m^{-1} \| r_m^{+}{}' \|^2 + m \| r_m^{+} \|^2) \, m \, | \hat{\Phi}_x^m(1)|^2$$

From (3.17), we derive as a consequence of [14]

$$| \Psi |_{2,\Omega}^2 \leqslant C (\sum_{m=-M}^{M} m \, | \hat{\Phi}_x^m(-1)|^2 + \sum_{m=-M}^{M} m \, | \hat{\Phi}_x^m(-1)|^2)$$
$$\leqslant C (\| \Phi_x(-1,\cdot) \|_{1/2,\Theta}^2 + \| \Phi_x(1,\cdot) \|_{1/2,\Theta}^2)$$
$$\leqslant C \| \Phi_x \|_{1,\Omega}^2 \, ,$$

so that

(3.19) $| \Psi |_{2,\Omega} \leqslant C \| \Phi \|_{2,\Omega}$.

Using $\Psi_x (\pm 1, 0) = 0$ and the Poincaré–Friedrichs inequality, we deduce that

(3.20) $\| \Psi_x \|_{1,\Omega} \leqslant C \| \Phi \|_{2,\Omega}$.

Besides, we note that $\Psi - \lambda y$ is a periodic function with zero average since Ψ^0 is equal to 0. Hence, it is standard to note that

$$\| \Psi - \lambda y \|_{0,\Omega} \leqslant C \| \Psi_y - \lambda \|_{0,\Omega} \leqslant C \| \Psi_{yy} \|_{0,\Omega} \leqslant C | \Psi |_{2,\Omega} \, ,$$

and from (3.15)(3.19)(3.20) we finally derive that

(3.21) $\| \Psi \|_2 \leqslant C \| \Phi \|_2$.

For the moment, we can notice that the function $\tilde{w}$ defined in (3.11) is not in X_h, but only in $[\mathbb{P}_N \otimes S_M] \times [(\mathbb{P}_N \cap H_0^1(\Lambda)) \otimes S_M]$. So, let us set $w = \tilde{w} + \text{curl } \Psi$, we note that

$$\text{div } w = \text{div } \tilde{w} + \text{div } (\text{curl } \Psi) = \Delta \Phi \, ,$$

and (3.8) is an easy consequence of (3.10) and (3.21).

Proof of lemma 3.1: case $K > 1$. Let us define the functions q_k in $L_0^2(\Omega_k)$ as follows

$$\forall (x,y) \in \Omega_k \, , \quad q_k(x,y) = q(x,y) - \alpha_k \, , \quad \alpha_k = \int_{\Omega_k} q(x,y) \, dxdy \, / \, \text{meas}(\Omega_k) \, .$$

Since we do not want any information for the values of q or div w on the virtual boundaries $a_k \times \Theta$, we simply construct K functions w_k on each Ω_k , $k = 1,\ldots,K$, as in the previous case such that

$$\forall k = 1,\ldots,K \, , \forall i = 1,\ldots,N-1 \, , \quad \text{div } w_k(\zeta_{i,k}, \cdot) = q_k(\zeta_{i,k}, \cdot) \, ,$$

$$\| w_k \|_{1,\Omega_k} \leqslant C \| q_k \|_{0,\Omega_k} \, .$$

Let us define the function $\overline{w}$ over Ω as follows

$$\forall \, x \in \Omega_k \, , \quad \overline{w}(x) = w_k(x) \, .$$

Due to the fact that $\mathbf{w}_k$ vanishes on the boundary of Ω_k, it is an easy matter to check that the function $\overline{\mathbf{w}}$ satisfies

$$\forall \; k = 1,...,K \; , \; \forall \; i = 1,...,N-1 \; , \quad \text{div}\, \overline{\mathbf{w}}\,(\zeta_{i,k}\,,.) = q(\zeta_{i,k}\,,.) - \alpha_k \; .$$

We only need now to add to $\overline{\mathbf{w}}$ a piecewise linear function to find the good solution. More precisely we define $\mathbf{w} = (w,z)$ by

$$\forall \; (x,y) \in \Omega_k \; , \quad \left| \begin{array}{l} w(x,y) = \overline{w}(x,y) + \alpha_k(x - a_k) + \sum_{\ell = 1}^{k-1} \alpha_\ell \, \text{meas}(\wedge_\ell) \\[2mm] z(x,y) = \overline{z}(x,y) \; . \end{array} \right.$$

It is an easy matter to check that w is still in $H^1_{0,\,*}(\Omega)$ since by hypothesis

$$2\pi \sum_{\ell = 1}^{K} \alpha_\ell \, \text{meas}(\wedge_\ell) = \int_\Omega q(x,y) \, dxdy = 0 \; .$$

The inequality (3.7) is also straightforward.

Remark 3.1 : We have used the hypothesis (3.1) in (3.34) and in the proof of the existence of functions satisfying (3.16)(3.17). In the general case, when (3.1) need not hold, if we work with more sophisticated combinations of the L'_n we can verify that (3.17) and (3.20) still holds with CM/N^2 in place of C in (3.20) and following the same lines as in the proof of the appendix, we can prove (3.10) with $C(M/N)(1+CM/N^2)$ in place of C. Furthermore, it follows from [4] that (3.8) follows in the general case, with CM in place of C.

3.2 Error estimate.

Let us first put the discrete problem in an abstract formulation, in order to apply the standard results of Brezzi [6] concerning the approximation of saddle-point problems like problem (2.10)(2.11) (see [10] or [5; sect.1] for more details and [1; sect.1] for a well-suited generalization for the numerical analysis of the Chebyshev spectral method).

Let us first define the bilinear forms

(3.22) $\quad \forall \; (\mathbf{u},\mathbf{w}) \in [C^1(\overline{\Omega})^2]^2 \; , \quad a_h(\mathbf{u},\mathbf{w}) = \nu(\nabla\mathbf{u},\nabla\mathbf{w})_{h,GL} \; ,$

(3.23) $\quad \forall \; (q,\mathbf{w}) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})^2 \; , \quad b_h(q,\mathbf{w}) = -(q,\text{div}\mathbf{w})_{h,G} \; .$

With these notations, problem (2.21)(2.22) can be rewritten as follows : *Find $(\mathbf{u}_h,p_h)$ in $X_h \times M_h$ such that*

(3.24) $a_h(\mathbf{u}_h,\mathbf{w}) + b_h(p_h,\mathbf{w}) = (\mathbf{f},\mathbf{w})_{h,GL}$, $\quad \forall \, \mathbf{w} \in X_h$,

(3.25) $b_h(q,\mathbf{u}_h) = 0$, $\quad \forall \, q \in M_h$.

The analysis of this problem will require four properties of a_h and b_h , that will be verified in the following lemmas.

<u>Lemma 3.2</u> : *There exist two constants* α *and* γ *independent of* h *such that*

(3.26) $\forall \, (\mathbf{u},\mathbf{w}) \in X_h^2$, $a_h(\mathbf{u},\mathbf{w}) \leqslant \gamma \, \| \mathbf{u} \|_1 \| \mathbf{w} \|_1$,

(3.27) $\forall \, \mathbf{u} \in X_h$, $a_h(\mathbf{u},\mathbf{u}) \geqslant \alpha \, \| \mathbf{u} \|_1^2$.

Proof : The case K = 1 is standard (see for instance [4; Prop. III.4]) and is based on the following inequalities (see [7; Lemma 3.2])

(3.28) $\forall \, (\varphi,\psi) \in \mathbb{P}_N \times \mathbb{P}_N$, $(\varphi,\psi)_{N,GL} \leqslant 3 \, \| \varphi \| \, \| \psi \|$,

(3.29) $\forall \, \varphi \in \mathbb{P}_N$, $(\varphi,\varphi)_{N,GL} \geqslant \| \varphi \|^2$.

We detail the analysis of the case K > 1. From $(2.20)_{GL}$ and (3.22) we have

$$\forall \, (\mathbf{u},\mathbf{w}) \in X_h \times X_h , \; a_h(\mathbf{u},\mathbf{w}) = (2\pi/(2M+1)) \sum_{k=1}^{K} \sum_{i=0}^{N} \sum_{j=0}^{2M} \nabla \mathbf{u}(\xi_{i,k},\theta_j) \nabla \mathbf{w}(\xi_{i,k},\theta_j) \varrho_{i,k}$$

from (2.18) we deduce

(3.30) $\forall \, (\mathbf{u},\mathbf{w}) \in X_h \times X_h$, $a_h(\mathbf{u},\mathbf{w}) = \int_\Theta dy \, [\sum_{k=1}^{K} \sum_{i=0}^{N} \nabla \mathbf{u}(\xi_{i,k},y) \nabla \mathbf{w}(\xi_{i,k},y) \varrho_{i,k}]$.

Using (3.28) we obtain now

$$\forall \, (\mathbf{u},\mathbf{w}) \in X_h \times X_h , \, a_h(\mathbf{u},\mathbf{w}) \leqslant 3 \int_\Theta dy \, [\sum_{k=1}^{K} \| \nabla \mathbf{u}(.,y) \|_{\Lambda_k} \| \nabla \mathbf{w}(.,y) \|_{\Lambda_k}] \; ,$$

$$\leqslant (3/2) \int_\Theta dy \, [\sum_{k=1}^{K} \| \nabla \mathbf{u}(.,y) \|_{\Lambda_k}^2]^{1/2} [\sum_{k=1}^{K} \| \nabla \mathbf{w}(.,y) \|_{\Lambda_k}^2]^{1/2} ,$$

$$\leqslant (3/2) \, (\int_\Theta \| \nabla \mathbf{u}(.,y) \|_{\Lambda}^2 \, dy)^{1/2} \, (\int_\Theta \| \nabla \mathbf{w}(.,y) \|_{\Lambda}^2 \, dy)^{1/2} \; ,$$

$$\leqslant (3/2) \, \| \mathbf{u} \|_1 \| \mathbf{w} \|_1 \; ,$$

which proves (3.26). From (3.29) and (3.30) we obtain (3.27) by using the same arguments.

Let us now analyze the properties of the discrete bilinear form b_h .

<u>Lemma 3.3</u> : *There exist two constants* β, δ, *independent of* h *such that*

(3.31) $\forall \, (q,\mathbf{w}) \in M_h \times X_h$, $b_h(q,\mathbf{w}) \leqslant \delta \, \| q \| \, \| \mathbf{w} \|_1$,

(3.32) $\quad \forall\, q \in M_h$, $\displaystyle\sup_{\mathbf{w}\,\in\,X_h} b_h(q,\mathbf{w}) \geqslant \beta\,\|q\|\,\|\mathbf{w}\|_1$.

Proof : From $(2.18)(2.20)_G$ and (3.23) we deduce

$$\forall\,(q,\mathbf{w}) \in M_h \times X_h\ ,\quad b_h(q,\mathbf{w}) = \int_\Theta dy\,[\textstyle\sum_{k=1}^{K} \sum_{i=1}^{N-1} q(\zeta_{i,k},y)\,\mathrm{div}\mathbf{w}(\zeta_{i,k},y)\,\omega_{i,k}\,]\ .$$

Using the Cauchy–Schwarz inequality gives for any $(q,\mathbf{w})$ in $M_h \times X_h$

$$b_h(q,\mathbf{w}) \leqslant \int_\Theta dy\,\sum_{k=1}^{K}\,([\textstyle\sum_{i=1}^{N-1} q^2(\zeta_{i,k},y)\omega_{i,k}]^{1/2}[\sum_{i=1}^{N-1}(\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\omega_{i,k}]^{1/2})\ ,$$

$$\leqslant [\textstyle\int_\Theta dy\,\sum_{k=1}^{K}\sum_{i=1}^{N-1} q^2(\zeta_{i,k},y)\omega_{i,k}]^{1/2}\,[\int_\Theta dy\,\sum_{k=1}^{K}\sum_{i=1}^{N-1}(\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\omega_{i,k}]^{1/2})\,,$$

Let us notice that $q^2(.,y)$ is in $\mathbb{P}_{2N-4}$; $(2.15)_G$ yields

(3.33) $\quad b_h(q,\mathbf{w}) \leqslant \|q\|\,[\textstyle\int_\Theta dy\,\sum_{k=1}^{K}\sum_{i=1}^{N-1}(\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\omega_{i,k}]^{1/2}$,

Writing now $\mathrm{div}\mathbf{w}$ in the Legendre basis over Λ_k , $k = 1,\dots,K$

(3.34) $\quad \forall\, x \in \Lambda_k$, $\forall\, y \in \Theta$, $k = 1,\dots,K$, $\mathrm{div}\mathbf{w}(x,y) = \sum_{n=0}^{N}\tau_{n,k}(y)\,L_n(x)$,

gives

$$\sum_{k=1}^{K}\sum_{i=1}^{N-1}(\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\omega_{i,k} = \sum_{k=1}^{K}\sum_{i=1}^{N-1}(\textstyle\sum_{n=0}^{N}\tau_{n,k}(y)\,L_n(\zeta_{i,k}))^2\,\omega_{i,k}$$

and finally

(3.35) $\quad \displaystyle\sum_{k=1}^{K}\sum_{i=1}^{N-1}(\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\,\omega_{i,k} = \sum_{k=1}^{K}\sum_{n=0}^{N}\sum_{\nu=0}^{N}(\tau_{n,k}\tau_{\nu,k})(y)\,[\textstyle\sum_{i=1}^{N-1}(L_n L_\nu)(\zeta_{i,k})\omega_{i,k}]$.

As a simple consequence of $(2.15)_G$ and (3.3) we deduce

(3.36) $\quad \forall\,(n,\nu)$, $n + \nu \leqslant 2N-3$, $\sum_{i=1}^{N-1}(L_n L_\nu)(\zeta_{i,k})\omega_{i,k} = (L_n\,,\,L_\nu) = \delta_{n\nu}(2/(2n+1))$.

Thanks to (2.14) and (3.5), we derive that

$$\forall\, i = 1,\dots,N-1\ ,\quad L_N(\zeta_{i,k}) = -((N-1)/N)\,L_{N-2}(\zeta_{i,k})\ ,$$

$$\forall\, i = 1,\dots,N-1\ ,\quad L_{N-1}(\zeta_{i,k}) = 0\ ,$$

whence, using again $(2.15)_G$ gives

$$\sum_{i=1}^{N-1}(L_N L_N)(\zeta_{i,k})\,\omega_{i,k} = ((N-1)/N)^2\,\sum_{i=1}^{N-1}(L_{N-2})^2(\zeta_{i,k})\,\omega_{i,k}$$

$$= ((N-1)/N)^2\,\int_\Lambda (L_{N-2})^2(x)\,dx$$

$$\leqslant 4/(2N+1) = 2\int_\Lambda (L_N)^2(x)\,dx\ ,$$

$$\sum_{i=1}^{N-1}(L_N L_{N-1})(\zeta_{i,k})\,\omega_{i,k} = 0\ ,$$

$$\sum_{i=1}^{N-1} (L_N L_{N-2})(\zeta_{i,k}) \, \omega_{i,k} = -((N-1)/N) \sum_{i=1}^{N-1} (L_{N-2})^2(\zeta_{i,k}) \, \omega_{i,k}$$

$$= -((N-1)/N) \int_\Lambda (L_{N-2})^2(x) \, dx$$

$$\leqslant 2 \int_\Lambda (L_N)^2(x) \, dx \ ,$$

$$\sum_{i=1}^{N-1} (L_{N-1} L_{N-1})(\zeta_{i,k}) \omega_{i,k} = 0 \ .$$

The four previous equalities together with (3.35) and (3.36) yield

$$\sum_{i=1}^{N-1} (\mathrm{div}\mathbf{w})^2(\zeta_{i,k},y)\omega_{i,k} \leqslant 3 \sum_{k=1}^{K} \sum_{n=0}^{N} \sum_{\nu=0}^{N} (\tau_{n,k}\tau_{\nu,k})(y) \, [\int_{\Lambda_k}(L_n L_\nu)(x) \, dx] ,$$

$$\leqslant 3 \int_\Lambda (\mathrm{div}\mathbf{w})^2(x,y) \, dx \ ,$$

which, from (3.33) gives (3.31) with $\delta = \sqrt{3}$.

Let us now prove that the compatibility condition (or inf-sup condition) between the discrete spaces M_h and X_h is satisfied. From Lemma 3.1 we know that there exists a function $\mathbf{w}$ in X_h such that

(3.37) $\forall \, k = 1,...,K \, , \, \forall \, i = 1,...,N-1 \, , \quad \mathrm{div}\,\mathbf{w}(\zeta_{i,k} \,,.) = q(\zeta_{i,k} \,,.) \ ,$

and

(3.38) $\| \mathbf{w} \|_1 \leqslant C \| q \| \ .$

With $(2.15)_G$ and (3.37) we easily verify that

$$b_h(q,\mathbf{w}) = (q,q)_{N,G} = \| q \|^2 \ ,$$

using now (3.38) yields

$$b_h(q,\mathbf{w}) \geqslant C \| q \| \, \| \mathbf{w} \|_1 \ .$$

The previous lemmas prove that the approximation of the Stokes problem by the scheme (2.21)(2.22) is well-posed. More precisely we obtain

<u>Theorem 3.1</u> : *For any* $\mathbf{f}$ *in* $(C^0_\#(\Omega))^2$ *, there exists a unique solution* $(\mathbf{u}_h \, , p_h)$ *to problem* (2.21)(2.22). *Moreover, if we assume that hypothesis* (3.1) *holds and that* $\mathbf{f}$ *belongs to* $H^\sigma_\#(\Omega)^2$ *,* $\sigma > 1$ *, the following error estimate for the velocity and the pressure holds for any* $\mu > 1/2$

(3.39) $\| \mathbf{u} - \mathbf{u}_h \|_1 + \| p - p_h \| \leqslant C \, (M^{-\sigma} + (1 + M^{-\mu}N^\mu)N^{1/2-\sigma}) \| \mathbf{f} \|_\sigma \ .$

Proof : The existence and uniqueness of $\mathbf{u}_h$ and p_h follows directly from Lemma 3.3 and 3.4 and [10, Theorem 1.1]. Moreover another consequence of that theorem is the following

$$(3.40) \quad \| \mathbf{u} - \mathbf{u}_h \|_1 + \| p - p_h \| \leqslant C [\inf_{\mathbf{w} \in X_h} \| \mathbf{u} - \mathbf{w} \|_1 + \inf_{q \in M_h} \| p - q \| + \sup_{\mathbf{w} \in X_h} \frac{(\mathbf{f},\mathbf{w}) - (\mathbf{f},\mathbf{w})_{h,GL}}{\| \mathbf{w} \|}],$$

and the result is an easy consequence of Theorem 2.1, Theorem B.5, Corollary B.1 and Theorem B.8 of the Appendix.

Remark 3.2 : Note that Theorem 3.1 still holds when $\mathbf{f}$ only satisfies

$$\mathbf{f}|_{\Omega_k} \in H^\sigma(\Omega_k) \ ,$$

see appendix B.

4. NUMERICAL RESULTS.

4.1 Implementation.

In this section we describe the details of the discrete equations (2.23) and (2.24). We start by defining the bases for the space $[\mathbb{P}_{N,K} \cap H_0^1(\Lambda)]^2 \times \mathbb{P}_{N-2,K}$ in which we search for our solution $(\mathbf{u}_N, p_N)$. As described in Section 2.2 Λ is divided into K spectral elements $\Lambda_1,...,\Lambda_K$. In each element Λ_k the velocity $\mathbf{u}_N$ of $[\mathbb{P}_{N,K} \cap H_0^1(\Lambda)]^2$ is expanded in terms of Nth order Lagrangian interpolants Q_i (see (2.27)) through the Legendre Gauss-Lobatto points ζ_i. We then define a mapping β from $x \in \Lambda_k$ onto $r \in I =]-1,1[$ as $r = -1 + 2(x - a_k)/(a_{k+1} - a_k)$. Then we state

$$(4.1) \qquad \forall \, x \in \Lambda_k \, , \quad \mathbf{u}_N^k(r,.) = \mathbf{u}_N(x,.)$$

and

$$(4.2) \qquad \mathbf{u}_N^k(r,.) = \sum_{i=0}^{N} \mathbf{u}_i^k Q_i(r) \, .$$

Here $\mathbf{u}_i^k = \mathbf{u}_N(\xi_{i,k} ,.)$ is the velocity at the (local) point $\xi_{i,k}$ in the interval Λ_k; that is, (4.1)(4.2) is a nodal basis. Similarly to the velocity the data $\mathbf{f}$ is also expanded in terms of Nth order Lagrangian interpolants through the Legendre Gauss-Lobatto points ξ_i ,

$$(4.3) \qquad \mathbf{f}_N^k(r) = \sum_{i=0}^{N} \mathbf{f}_i^k Q_i(r) \, .$$

The pressure $p_N \in \mathbb{P}_{N-2,K}$ is expanded in terms of $(N-2)^{th}$ order Lagrangian interpolants $\tilde{Q}_i$ through the Legendre Gauss points ζ_i ,

$$(4.4) \qquad p_N^k(r) = \sum_{i=1}^{N-1} p_i^k \tilde{Q}_i$$

where $p_i^k = p_N(\zeta_{i,k})$ is pressure at the (local) point $\zeta_{i,k}$ in the interval Λ_k. Note that the Gauss points are naturally suited for the pressure, which need not be continuous across elemental boundaries.

The expansions (4.1)–(4.3) are now inserted into (2.23) and (2.24), and the discrete equations are generated by choosing test functions $\mathbf{w} \in [\mathbb{P}_{N,K} \cap H_0^1(\Lambda)]^2$ in (2.23) which are unity at a single $\zeta_{i,k}$ and zero at all other Legendre Gauss-Lobatto points, and test functions $q \in \mathbb{P}_{N-2,K}$ in (2.24) which are unity at a single $\zeta_{i,k}$ and zero at all other Legendre Gauss points. To evaluate the integrals in (2.23) and (2.24) we use numerical quadrature through the Legendre Gauss-Lobatto points $\xi_{i,k}$ and the Gauss points $\zeta_{i,k}$, denoted $(\, . \, , \, . \,)_{N,GL}$ and $(\, . \, , \, . \,)_{N,G}$

respectively. The first term in (2.23) can then be written as

(4.5) $\qquad (u_{Nx}, w_x)_{N,GL} = \sum_{k=1}^{'K} (L_k /2) \sum_{j=0}^{N} \sum_{m=0}^{N} \varrho_m D_{mi} D_{mj} u_j^k$,

where the derivative matrix D is defined as

(4.6) $\qquad D_{pq} = (dQ_q / dr)(\xi_q)$.

We recall that ϱ_m are the quadrature weights associated with the Legendre Gauss–Lobatto points ξ_i , also $L_k = (a_{k+1} - a_k)$ and $\sum'$ denotes direct stiffness summation.

The second term in (2.23) becomes

(4.7) $\qquad (u_N, w)_{N,GL} = \sum_{k=1}^{'K} (L_k /2) \varrho_i u_i^k$,

while the right–hand side of (2.23) can be written as

(4.8) $\qquad (f, w)_{N,GL} = \sum_{k=1}^{'K} (L_k /2) \varrho_i f_i^k$.

The left–hand side of (2.24) becomes

(4.9) $\qquad (q, u_{Nx} + inv_N)_{N,G} = \omega_i \left(\sum_j \tilde{D}_{ij} u_j^k + in \, L_k/2 \, \tilde{I}_{ij} v_j^k \right)$

where the derivative matrix $\tilde{D}$ and interpolation operator $\tilde{I}$ are defined as

(4.10) $\qquad \tilde{D}_{pq} = dQ_q/dr (\zeta_p)$,

(4.11) $\qquad \tilde{I}_{pq} = Q_q(\zeta_p)$.

Note that in (4.6) no direct stiffness summation need be performed since the Legendre Gauss points $\zeta_{i,k}$ are all distinct.

In matrix form the set of discrete equations (2.23) and (2.24) can be written as

(4.12) $\qquad \mathbf{A}_{app} \mathbf{u} - \mathbf{G}_{app} p \, \square \, \mathbf{B} \, f$

(4.13) $\qquad \mathbf{D}_{app} \mathbf{u} = 0$

where $\mathbf{A}_{app}$ is given by (4.5)–(4.7) , $\mathbf{B}_{app}$ by (4.8) , $\mathbf{D}_{app}$ by (4.9) , and $\mathbf{G}_{app}$ is the adjoint–matrix of $\mathbf{D}_{app}$. The Uzawa method used to solve (4.12)–(4.13) will be described in more details in a future paper[19]. Basically, it consists in solving the following zero[th]–order equation for the pressure

$$\mathbf{D}_{app} (\mathbf{A}_{app})^{-1} \mathbf{G}_{app} p = \mathbf{D}_{app} (\mathbf{A}_{app})^{-1} \mathbf{B} \, f \, ,$$

by using a conjugate gradient algorithm, and then recovering the velocity from (4.12).

4.2. Numerical Results.

In this section we discuss some numerical results obtained by solving the set of discrete equations (4.12)(4.13). Two test problems have been solved, both in which $v = 1$ and $n = 1$. The solution $(\mathbf{u}, p)$ and the data $\mathbf{f} = (f, g)$ for the first test problem are

$$(4.14) \quad \begin{cases} u = -(1 + \cos \pi x)/\pi \ , \\ v = i \sin \pi x \ , \\ p = \sin \pi x \ , \\ f = -(1 + \cos \pi x)/\pi \ , \\ g = i(2+\pi^2) \sin \pi x \ , \end{cases}$$

while the solution and the data for the second test problem are

$$(4.15) \quad \begin{cases} u = -(1 + \cos \pi x)/\pi \ , \\ v = i \sin \pi x \ , \\ p = \sin \pi x \ |x-1/2|^{\gamma+2/3} \ , \\ f = [-(1 + \cos \pi x)/\pi + \gamma+2/3) \ \text{sgn}(x-1/2) \ |x-1/2|^{\gamma+2/3} \ ] \ , \\ g = i[(2+\pi^2) \sin \pi x + |x-1/2|^{\gamma+2/3} \ ] \ . \end{cases}$$

Note that the solution and the data in the first test problem are infinitely smooth, while the regularity of second test problem is determined by the value of γ, which is assumed to be an integer.

In the first test problem Λ is divided into 2 equal subintervals Λ_1 and Λ_2, i.e. $k = 2$, while in the second test problem only one element is used, i.e. $K = 1$. The numerical solutions are compared with the analytical solutions for different values of N, the order of the polynomial expansions (4.1)(4.3). To measure the error in the numerical solutions, the following error measures are used:

$$(4.16) \quad \|\mathbf{u}-\mathbf{u}_N\|_{1,GL} = \{\textstyle\sum_{k=1}^{K} (L_k/2) [\sum_{i=0}^{N} \rho_i (\sum_{j=0}^{N} [D_{ij}(\mathbf{u}(\xi_{j,k})-\mathbf{u}_j^k)]^2 + [\mathbf{u}(\xi_{i,k})-\mathbf{u}_i^k]^2)]\}^{1/2}$$

$$(4.17) \quad \|p-p_N\|_{0,G} = \{\textstyle\sum_{k=1}^{K} (L_k/2) (\sum_{i=1}^{N-1} \omega_i [p(\zeta_{i,k})-p_i^k]^2)\}^{1/2} \ .$$

In the first test problem we obtain exponential convergence as the order N of the polynomial

expensions is increased. Figure 1 shows the error in the velocity and the pressure as a function of the total number of degrees of freedom (Legendre Gauss Lobatto points) in the x-direction. The rapid convergence rate is expected due to the fact that the solution is analytic.

In the second test problem we obtain algebraic convergence as the order N of the polynomial expansions is increased. Figure 2 shows the error in the pressure as a function of the total number of degrees of freedom (Legendre Gauss Lobatto points) in the x-direction for $\gamma = 3$ and $\gamma = 5$. The convergence rate is given approximately as $N^{-(\gamma+1)}$. Although the error estimates (3.40) is somewhat pessimistic as regards the error due to the forcing term $(f \in H^{\gamma} \Rightarrow \|p-p_N\| < N^{1-\gamma})$, as regards the approximation errors $(p \in H^{\gamma+1} \Rightarrow \|p-q_N\| < N^{-1-\gamma})$ the bound is quite tight.

APPENDIX.

Appendix A - Approximation of a Discrete Laplace Equation.

The proof of the compatibility condition between the spaces of velocity and pressure involves some results concerning the approximation of the solution of the Laplace equation by a collocation method based on the Gauss points.

Lemma A.1 : *For any* q *in* M_h *there exists a unique* Φ *in* $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ *such that*

(A.1) $\qquad \forall \ i = 1,...,N-1 \ , \quad \Delta\Phi(\zeta_i,.) = q(\zeta_i,.)$.

Proof : Let us consider the collocation problem : *Find* Φ *in* $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ *such that*

(A.2) $\qquad \forall \ i = 1,...,N-1 \ , \ \forall \ j = 0,...,2M \ , \quad \Delta\Phi(\zeta_i,\theta_j) = q(\zeta_i,\theta_j).$

It is an easy matter to check that Φ satisfies (A.1). Multiplying both sides of (A.2) by $(2\pi/(2M+1))\,\Psi(\zeta_i,\theta_j)\omega_i$ and summing up with respect to i and j leads to the equation

(A.3) $\qquad \forall \ \Psi \in \mathbb{P}_N \otimes S_M \ , \quad (\Delta\Phi,\Psi)_{h,G} = (q,\Psi)_{h,G}$.

Taking Ψ equal to the Lagrange interpolant of the point (ζ_i,θ_j) in $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$, proves that the problem (A.2) is equivalent to : *Find* Φ *in* $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ *such that*

(A.4) $\qquad \forall \ \Psi \in [\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M \ , \quad (\Delta\Phi,\Psi)_{h,G} = (q,\Psi)_{h,G}$.

Let us set

(A.5) $\qquad \forall \ (\Psi,X) \in \{[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M\}^2 \ , \quad c(\Psi,X) = -(\Delta\Psi,X)_{h,G}$

In order to prove that c is continuous and elliptic, we recall that (see [2; Lemma III.1])

(A.6) $\qquad \forall \ \psi \in \mathbb{P}_{N-2} \ , \ \int_\Lambda (1-x^2)\,\psi(x)^2 \,dx \leqslant ((1-x^2)\,\psi,\psi)_{N,G} \leqslant 2 \int_\Lambda (1-x^2)\,\psi(x)^2 \,dx$,

and that (see [2; Lemma III.2])

(A.7) $\qquad \forall \ \psi \in \mathbb{P}_{N-2} \ , \ C\,N^{-1}\int_\Lambda (1-x^2)^2\psi(x)^2 \,dx \leqslant ((1-x^2)^2\psi,\psi)_{N,G} \leqslant C \int_\Lambda (1-x^2)^2\psi(x)^2 \,dx$.

For any Ψ and X in $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ we derive from $(2.15)_G$ and (2.18) that

(A.8) $\qquad c(\Psi,X) = \int_\Theta [-(\Psi_{xx}(.,y),X(.,y))_{N,G} + (\Psi_y(.,y),X_y(.,y))_{N,G}]\,dy$.

Let us write $\Psi(.,y)$ and $X(.,y)$ in the basis $(1-x^2) L_n'$, $n = 1,...,N-1$

$$\Psi(x,y) = \sum_{n=1}^{N-1} \psi_n(y) (1-x^2) L_n'(x) \ ,$$

$$X(x,y) = \sum_{n=1}^{N-1} \chi_n(y) (1-x^2) L_n'(x) \ .$$

Using (3.4) gives

$$-(\Psi_{xx}(.,y),X(.,y))_{N,G} = \sum_{n=1}^{N-1} \sum_{\ell=1}^{N-1} \psi_n(y) \chi_\ell(y) \ell(\ell+1) ((1-x^2)L_n',L_\ell')_{N,G} \ .$$

From $(2.15)_G$ (3.3)(3.4) and (A.6) we deduce

(A.9) $\quad -(\Psi_{xx}(.,y),X(.,y))_{N,G} \leqslant (4/(2n+1)) \sum_{n=1}^{N-1} \psi_n(y)\chi_n(y) (n(n+1))^2 \leqslant 2 \int_\Lambda \Psi_x X_x (x,y) \, dx$

and that

(A.10) $\quad -(\Psi_{xx}(.,y),\Psi(.,y))_{N,G} \geqslant (2/(2n+1))\sum_{n=1}^{N-1} \psi_n(y)^2 (n(n+1))^2 \geqslant -\int_\Lambda \Psi_{xx}(x,y)\Psi(x,y) \, dx$

Let us write now

$$\Psi_y(x,y) = (1-x^2) \tilde{\Psi}(x,y) \ ,$$

$$X_y(x,y) = (1-x^2) \tilde{X}(x,y) \ ,$$

using (A.7) yields to

(A.11) $\quad (\Psi_y(.,y),X_y(.,y))_{N,G} \leqslant C \ (\Psi_y(.,y),X_y(.,y)) \ ,$

and to

(A.12) $\quad (\Psi_y(.,y),\Psi_y(.,y))_{N,G} \geqslant C N^{-1} (\Psi_y(.,y),\Psi_y(.,y)) \ ,$

Finally, due to (A.8)(A.9)(A.11) and the Poincaré-Friedrichs inequality, we deduce that c is uniformly continuous over $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$. Due to (A.8)(A.10)(A.12), we derive a (nonuniform) ellipticity of c over $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$, more precisely, we obtain

(A.13) $\quad \forall \Psi \in [\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M \ , \ c(\Psi,\Psi) \geqslant C N^{-1} \| \Psi \|_1^2 \ .$

From the Lax-Milgram lemma we conclude that problem (A.4) is well-posed, i.e. (A.4) admits a unique solution Φ in $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$. Besides from (A.13) we derive the following estimate

(A.14) $\quad \| \Phi \|_1 \leqslant C N^{-1} \| q \| \ .$

In this second lemma, we are going to derive a uniform bound for the $H^1(\Omega)$-norm of Φ_x .

Lemma A.2 : *The solution Φ in $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ of problem (A.1) satisfies the following estimate*

(A.15) $\quad \| \Phi_x \|_1 \leqslant C \| q \| \ .$

Proof : Let us choose now $\Psi = \Phi_{xx}$ in (A.3), we derive

$$(\Phi_{xx}, \Phi_{xx})_{h,G} + (\Phi_{yy}, \Phi_{xx})_{h,G} = (q, \Phi_{xx})_{h,G} \ ,$$

and we obtain from $(2.15)_G$, (2.19) and $(2.20)_G$ that

(A.16) $\quad \| \Phi_{xx} \|^2 - \int_\Theta (\Phi_y, \Phi_{xxy})_{N,G} \leqslant \| q \| \| \Phi_{xx} \|$.

Writing $\Phi_y(.,y)$ in the basis $(1-x^2) L_n'$, $n = 1, ..., N-1$,

(A.17) $\quad \forall (x,y) \in \Omega, \ \Phi_y(x,y) = \sum_{n=1}^{N-1} \varphi_n(y) (1-x^2) L_n'(x)$, $\quad \varphi_n \in S_M$,

gives, as in (A.10),

(A.18) $\quad -(\Phi_y, \Phi_{xxy})_{N,G} \geqslant (2/(2n+1)) \sum_{n=1}^{N-1} \varphi_n(y)^2 (n(n+1))^2 \geqslant \int_\Lambda \Phi_{xy}^2 (x,y) \, dx$.

Finally, we conclude with (A.16) that

(A.19) $\quad \| \Phi_x \|_1 \leqslant 2 \| q \|$.

Unfortunately, the inequality $\| \Phi_{yy} \| \leqslant C \| q \|$ is not so easy to derive. This is done is the following lemma.

<u>Lemma A.3</u> : *The solution* Φ *in* $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ *of problem* (A.1) *satisfies the following estimate*

(A.20) $\quad \| \Phi_y \|_1 \leqslant C \| q \|$.

Proof : We first define a function $\tilde{\Phi}$ such that

(A.21) $\quad \forall i = 1, ..., N-1$, $\tilde{\Phi}(\zeta_i, .) = \Phi(\zeta_i, .)$,

and such that the inequality $\| \tilde{\Phi}_{yy} \| \leqslant C \| q \|$ holds. To this end we use the basis $(1-x^2) L_n''$, $n = 2, ..., N$,

(A.22) $\quad \forall (x,y) \in \Omega, \ \Phi(x,y) = \sum_{n=2}^{N} \eta_n(y) (1-x^2) L_n''(x)$, $\quad \eta_n \in S_N$,

From (3.4), (3.5) and (3.6) we obtain that

(A.23) $\quad \begin{cases} (1-x^2) L_N''(x) = (1-x^2) L_{N-2}''(x) - N(2N-1) L_N(x) + N(2N-1) x L_{N-1}(x) \ , \\ (1-x^2) L_{N-1}''(x) = 2 x L_{N-1}'(x) + N(N-1) L_{N-1}(x) \ , \end{cases}$

so that

$$\forall i = 1, ..., N-1 \ , \ (1-\zeta_i^2) L_N''(\zeta_i) = (1-x^2) L_{N-2}''(\zeta_i) - N(2N-1) L_N(\zeta_i) \ ,$$

$$\forall\ i = 1,...,N-1\ ,\quad (1-\zeta_i^2)L_{N-1}''(\zeta_i) = 2\zeta_i L_{N-1}'(\zeta_i)\ .$$

Hence the function $\tilde{\Phi}$ defined by

$$(A.24)\quad \forall\ (x,y) \in \Omega,\ \tilde{\Phi}(x,y) = \sum_{n=2}^{N-2} \eta_n(y)(1-x^2)L_n''(x) + 2\eta_{N-1}(y)\,x L_{N-1}'(x)$$
$$+\eta_N(y)[(1-x^2)L_{N-2}''(x) - N(2N-1)L_N(x)]\ ,$$

satisfies (A.21).

Moreover we have

$$(A.25)\quad \forall\ (x,y) \in \Omega,\ \tilde{\Phi}(x,y) = \Phi(x,y) - \eta_{N-1}(y)\,(N(N-1))\,L_{N-1}(x)$$
$$- \eta_N(y)\,(N(2N-1))\,x\,L_{N-1}(x)\ .$$

Using now (A.3) and (A.21) gives

$$(\Phi_{xx},\tilde{\Phi}_{yy})_{h,G} + (\tilde{\Phi}_{yy},\tilde{\Phi}_{yy})_{h,G} = (q,\tilde{\Phi}_{yy})_{h,G}\ ,$$

so that as previously from (A.18)

$$(A.26)\quad \|\Phi_{xy}\|^2 + (\tilde{\Phi}_{yy},\tilde{\Phi}_{yy})_{h,G} \leqslant \|q\|^2\ .$$

From (A.24), we obtain

$$(A.27)\quad \tilde{\Phi}_{yy}(x,y) = \sum_{n=2}^{N-3} \eta_n''(y)(1-x^2)L_n''(x) + [(\eta_{N-2}''(y)+\eta_N''(y)](1-x^2)L_{N-2}''(x)$$
$$+2\eta_{N-1}''(y)\,x\,L_{N-1}'(x) - \eta_N''(y)(N(2N-1))L_N(x)\ .$$

Using now the formula derived from (3.3), (3.4) and (3.6) yields for any $n \geqslant \nu$

$$\int_\Lambda (1-x^2)^2 L_n''(x)L_\nu''(x)\,dx = \int_\Lambda (2x\,L_n'(x) - (n(n+1))L_n(x))(1-x^2)L_\nu''(x)\,dx$$
$$= \int_\Lambda [((1-x^2)L_n'(x))(2xL_\nu''(x)) - (n(n+1))L_n(x)(1-x^2)L_\nu''(x)]\,dx$$
$$= \int_\Lambda -(n(n+1)/(2n+1))(L_{n+1}(x) - L_{n-1}(x))(2xL_\nu''(x))\,dx$$
$$- \int_\Lambda (n(n+1))L_n(x)(1-x^2)L_\nu''(x)\,dx$$
$$= \int_\Lambda (n(n+1)/(2n+1))(L_{n-1}(x))(2xL_\nu''(x))\,dx$$
$$- \int_\Lambda (n(n+1))L_n(x)(1-x^2)L_\nu''(x)\,dx$$
$$= \lambda_n\,\delta_{n\nu}\ .$$

To derive the value of λ_n we only calculate the leading coefficient of $2xL_n''(x)$ and $(1-x^2)L_n''(x)$, this is done by using (3.5), we obtain

$$\lambda_n = \int_\Lambda (n(n+1)/(2n+1))(L_{n-1}(x))(2(2n-1)(n-1)L_{n-1}(x) + ...)$$
$$+ (n(n+1))L_n(x)(n(n-1)L_n(x) + ...)\,dx\ ;$$

and by (3.3),

$$\lambda_n = 2\,(n-1)n(n+1)(n+2)/(2n+1)\ .$$

The case $n < \nu$ is treated in a symetric way. We can write, for any n and ν in $\mathbb{N}$

$$\int_\Lambda (1-x^2)^2\, L_n''(x)L_\nu''(x)\, dx = \lambda_n\, \delta_{n\nu}\ .$$

We compute from (A.23) and (A.27) on the one hand

$$(A.28) \quad \|\tilde{\Phi}_{yy}\|^2 = \sum_{n=2}^{N-3} \|\eta_n''\|^2\, \lambda_n + \|\eta_{N-2}'' + \eta_N''\|^2\lambda_{N-2} + 4\|\eta_{N-1}''\|^2\, \|xL_{N-1}'\|^2$$
$$+ \|\eta_N''\|^2\, 2[N(2N-1)]^2(1/(2N+1))\ ,$$

and on the other hand from $(2.15)_G$

$$(A.29) \quad (\tilde{\Phi}_{yy},\tilde{\Phi}_{yy})_{h,G} = \sum_{n=2}^{N-3} \|\eta_n''\|^2\, \lambda_n + \|\eta_{N-2}'' + \eta_N''\|^2\lambda_{N-2} + 4\|\eta_{N-1}''\|^2\, (x^2 L_{N-1}',L_{N-1}')_{N,G}$$
$$+ \|\eta_N''\|^2\, [N(2N-1)]^2(L_N,L_N)_{N,G}$$
$$- 2\,(N(2N-1))\,(\eta_{N-2}'' + \eta_N'',\eta_N'')_\Theta\,((1-x^2)\,L_{N-2}'',L_N)_{N,G}\ .$$

From $(2.15)_G$, (3.3) and (3.5) we deduce

$$(A.30) \quad (L_N,L_N)_{N,G} = (1-N^{-1})^2\,(L_{N-2},L_{N-2})_{N,G} = (1-N^{-1})^2\,(L_{N-2},L_{N-2}) \geqslant 1/(2N+1)\ .$$

Using now (3.6) combined with (3.5), we compute

$$xL_{N-1}' = (N-1)\,L_{N-1} + (2N-5)\,L_{N-3} + (N-4)\,L_{N-5} + xL_{N-5}'\ ,$$

so that

$$(x^2 L_{N-1}',L_{N-1}')_{N,G} = (2N-5)^2\,[2/(2N-5)] + \|(N-4)\,L_{N-5} + xL_{N-5}'\|^2$$

and

$$\|xL_{N-1}'\|^2 = (N-1)^2\,[2/(2N-1)] + (2N-5)^2\,[2/(2N-5)] + \|(N-4)\,L_{N-5} + xL_{N-5}'\|^2\ .$$

We conclude with

$$(A.31) \quad \|xL_{N-1}'\|^2 \leqslant 2\,(x^2 L_{N-1}',L_{N-1}')_{N,G}\ .$$

Finally, from (3.5) and $(2.15)_G$, we derive that

$$((1-x^2)\,L_{N-2}'',L_N)_{N,G} = ((N-1)/N)\,((1-x^2)\,L_{N-2}'',L_{N-2})_{N,G}$$
$$= ((N-1)/N)\,((1-x^2)\,L_{N-2}'',L_{N-2})\ .$$

Next, using (3.3), we obtain from the equality $(1-x^2)\,L_{N-2}'' = -(N-2)(N-3)\,L_{N-2} + \ldots$

$$(A.32) \quad ((1-x^2)\,L_{N-2}'',L_N)_{N,G} = -2(N-2)(N-3)(N-1)/N(2N-3) = \mathcal{O}(N)$$

Recalling now (A.26)(A.28)–(A.30), we prove that

$$(A.33) \quad \|\tilde{\Phi}_{yy}\| \leqslant C\,(\|q\| + N^{3/2}\,\|\eta_{N-2}'' + \eta_N''\| + N^{3/2}\,\|\eta_N''\|)$$

Formula (3.6) combined with (A.17) and (A.22) yields

(A.34) $\quad \begin{cases} \eta'_{N-2}(y) + \eta'_N(y) = \varphi_{N-3}(y)/(2N-5) \ , \\ \eta'_{N-1}(y) = \varphi_{N-2}(y)/(2N-3) \ , \\ \eta'_N(y) = \varphi_{N-1}(y)/(2N-1) \ . \end{cases}$

In order to estimate $\| \Phi_{yy} \|$ we note that, from (3.3)(3.6) and (A.25) we obtain

$$\| \Phi_{yy} \| \leqslant C \, [\| \tilde{\Phi}_{yy} \| + N^{3/2} \| \eta''_{N-1} \| + N^{3/2} \| \eta''_N \|] \ ,$$

and from (A.33)(A.34) we deduce

$$\| \Phi_{yy} \| \leqslant C \, [\| q \| + N^{1/2} \, [\| \varphi'_{N-3} \| + \| \varphi'_{N-2} \| + \| \varphi'_{N-1} \|] \ .$$

Let us now use the inverse inequality

$$\forall \, (r,s) \in \mathbb{R}_+^2 \ , \ 0 \leqslant r \leqslant s \ , \ \forall \, \chi \in S_M \ , \ \| \chi \|_s \leqslant C \, M^{s-r} \| \chi \|_r \ ,$$

it follows that

$$\| \Phi_{yy} \| \leqslant C \, [\| q \| + M(\, N^{1/2} \, [\| \varphi_{N-3} \| + \| \varphi_{N-2} \| + \| \varphi_{N-1} \|])] \ .$$

It is an easy consequence of (A.17) that

$$\sum_{n=1}^{N-1} \varphi_n(y)^2 \, (n(n+1))^2 \, 2/(2n+1) \leqslant \int_\Lambda \Phi_{xy}^2 \, (x,y) \, dx \ ,$$

so that, from (A.19) we derive

$$N^{3/2} \, [\| \varphi_{N-3} \| + \| \varphi_{N-2} \| + \| \varphi_{N-1} \|] \leqslant C \| q \| \ ,$$

hence, we conclude

(A.35) $\quad \| \Phi_{yy} \| \leqslant C(1+M/N) \| q \| \ ,$

which, thanks to (3.1) and (A.19), finishes the proof of Lemma A.3 .

As a conclusion, we state

<u>Theorem A.1</u>: *For any* q *in* M_h *there exists a unique* Φ *in* $[\mathbb{P}_N \cap H_0^1(\Lambda)] \otimes S_M$ *such that*

$$\forall \, i = 1,...,N-1 \ , \ \Delta\Phi(\zeta_i,.) = q(\zeta_i,.) \ .$$

Moreover, Φ *satisfies the following bound*

(A.36) $\quad \| \Phi \|_2 \leqslant C \| q \|.$

Appendix B- Error Bound for the Projection and Interpolation Operators.

The final error estimates require some technical results about the orthogonal projection operator for the $H^1_{0,*}(\Omega)$-scalar product onto the space $\mathbb{P}_{N,K} \otimes S_M$ and interpolation operators with values in the same space.

B.1 The one-dimensional case.

In this paragraph, we extend the results of [7] and [16] to state some properties of the approximation operators over $\mathbb{P}_{N,K}$. See also [9] for some partial results in this direction.

The possibility of using different values of the parameter N in each subdomain Ω_k was only evoqued in the previous part of the numerical analysis. The only difficulty that this should imply, would be to complicate the reading of the proofs, and absolutely not of mathematical nature. Howether, here we shall consider such an eventuality since the extension from the case where N is assumed to be constant, to the one where N is variable, is not straightforward. The interest of doing so is to be able to fit the regularity of the solution and, in particular, to increase the number of degree in the region where the solution is a bit less regular. This is a first step toward the general situation; the second one will take into account the size of Λ_k and the possibility of taking the parameter K as a discretization parameter.

Let us define for each k, $1 \leqslant k \leqslant K$, an integer N(k), that will take now the place of the previous notion of N, the degree of the polynomial in the nonperiodic direction; the corresponding space of polynomials over Λ will be noted $\tilde{\mathbb{P}}_{N,K}(\Lambda)$ and will consist in

$$\tilde{\mathbb{P}}_{N,K}(\Lambda) = \{ \varphi \in L^2(\Lambda), \; \varphi_{|\Lambda_k} \in \mathbb{P}_{N(k)}(\Lambda_k) \}$$

(note that from now on , we shall precise the interval where the variable are defined for the various spaces of polynomials).

The regularity of the solution being possibly different on each Λ_k, we introduce some spaces with broken norms. More precisely, for any K-tuple of positive real numbers $\mathbf{r} = (r_1, r_2,...,r_K)$, we

define

$$\tilde{H}^r(\Lambda) = \{ \varphi \in L^2(\Lambda), \ \varphi_{|\Lambda_k} \in H^{r_k}(\Lambda_k) \} \ ,$$

this space being equipped with the norm

$$\forall \ \varphi \in \tilde{H}^r(\Lambda) \ , \ \| \varphi \|_{r,\Lambda} = [\sum_{k=1}^{K} \| \varphi \|_{r_k,\Lambda_k}^2]^{1/2} \ .$$

First of all, let us consider the $L^2(\Lambda)$-projection operator $\tilde{\Pi}_N$ onto $\tilde{\mathbb{P}}_{N,K}(\Lambda)$. We have

Theorem B.1 : *Let* **r** *be a K-tuple of nonnegative real numbers. We have for any function* Ψ *in* $\tilde{H}^r(\Lambda)$

(B.1) $\quad \| \Psi - \tilde{\Pi}_N \Psi \|_{0,\Lambda} \leqslant C [\sum_{k=1}^{K} N(k)^{-2r_k} \| \Psi \|_{r_k,\Lambda_k}^2]^{1/2} \ .$

Proof : First, we note that from the definition of $\tilde{\Pi}_N$, we have

(B.2) $\quad \forall \ \Psi \in L^2(\Lambda) \ , \ \forall \ \Phi \in \tilde{\mathbb{P}}_{N,K}(\Lambda) \ , \ (\Psi - \tilde{\Pi}_N \Psi, \Phi)_\Lambda = 0 \ ,$

so that for any k in $\mathbb{N}$, $1 \leqslant k \leqslant K$, we have

$$\forall \ \Psi \in L^2(\Lambda) \ , \ \forall \ \Phi \in \mathbb{P}_{N(k)}(\Lambda_k) \ , \ \int_{\Lambda_k} (\Psi - \tilde{\Pi}_N \Psi)_{|\Lambda_k} \Phi \ dx = 0 \ ,$$

hence we note that $(\tilde{\Pi}_N \Psi)_{|\Lambda_k}$ is the projection of $\Psi_{|\Lambda_k}$ onto $\mathbb{P}_{N(k)}(\Lambda_k)$ with respect to the $L^2(\Lambda_k)$-scalar product. As a consequence of classical results (see [7; Theorem 2.3]), we derive that for any $r_k \geqslant 0$, we have

$$\forall \ \Psi \in H^{r_k}(\Lambda_k) \ , \ \| \Psi - \tilde{\Pi}_N \Psi \|_{0,\Lambda_k} \leqslant C \ N(k)^{-r_k} \| \Psi \|_{r_k,\Lambda_k} \ .$$

Summing up the square of these inequalities, we deduce that

$$\forall \ \Psi \in \tilde{H}^r(\Lambda) \ , \ \| \Psi - \tilde{\Pi}_N \Psi \|_{0,\Lambda}^2 = \sum_{k=1}^{K} \| \Psi - \tilde{\Pi}_N \Psi \|_{0,\Lambda_k}^2$$

$$\leqslant C \sum_{k=1}^{K} N(k)^{-2r_k} \| \Psi \|_{r_k,\Lambda_k}^2 \ ,$$

and (B.1) is proved.

Next, we state the following inverse inequality

<u>Lemma B.1</u> : *Let* **r** *and* **s** *be two* K−*tuples of real numbers, such that for any* k, $1 \leqslant k \leqslant K$, $0 \leqslant r_k \leqslant s_k$. *We have for any any function* Ψ *in* $\tilde{\mathbb{P}}_{N,K}(\Lambda)$

(B.3) $\|\Psi\|_{s,\Lambda} \leqslant C [\sum_{k=1}^{K} N(k)^{4(s_k - r_k)} \|\Psi\|_{r_k,\Lambda_k}^2]^{1/2}$.

Proof : Here again this result is a simple consequence of the following classical inverse inequality over $\mathbb{P}_N(]a,b[)$ for any a and b in $\mathbb{R}$:

(B.4) $\forall w \in \mathbb{P}_N(]a,b[)$, $\forall(\varrho,\sigma) \in \mathbb{R}_+^2$, $\varrho \leqslant \sigma$, $\|w\|_{\sigma,]a,b[} \leqslant C N^{2(\sigma-\varrho)} \|w\|_{\varrho,]a,b[}$.

Indeed, we have

$\forall v \in \tilde{\mathbb{P}}_{N,K}(\Lambda)$, $\|v\|_{s,\Lambda}^2 = \sum_{k=1}^{K} \|v\|_{s_k,\Lambda_k}^2$,

from (B.4) applied on each subinterval Λ_k , we deduce that

$\forall v \in \tilde{\mathbb{P}}_{N,K}(\Lambda)$, $\|v\|_{s,\Lambda}^2 \leqslant C \sum_{k=1}^{K} N^{4(s_k - r_k)} \|v\|_{r_k,\Lambda_k}^2$,

and (B.3) is proved.

Now, we are interested in some projection operator from $H_0^1(\Lambda)$ onto $\tilde{\mathbb{P}}_{N,K}(\Lambda) \cap H_0^1(\Lambda)$.

<u>Theorem B.2</u> : *There exists an operator* $\tilde{\pi}_N^1$ *from* $H_0^1(\Lambda)$ *onto* $\tilde{\mathbb{P}}_{N,K}(\Lambda) \cap H_0^1(\Lambda)$ *verifying for any function* Ψ *in* $\tilde{H}^s(\Lambda) \cap H_0^1(\Lambda)$, *with* **s** *being one* K−*tuples of real numbers* $\geqslant 1$

(B.5) $\forall \mathbf{r} = (r_k)_{1 \leqslant k \leqslant K}$, $0 \leqslant r_k \leqslant 1$, $\|\Psi - \tilde{\pi}_N^1 \Psi\|_{r,\Lambda} \leqslant C [\sum_{k=1}^{K} N(k)^{2(r_k - s_k)} \|\Psi\|_{s_k,\Lambda_k}^2]^{1/2}$.

Proof : Let us recall that, for any a and b in $\mathbb{R}$, there exists a projection operator π_N from $H^1(]a,b[)$ onto $\mathbb{P}_N(]a,b[)$ satisfying (see [3; Corollary IV.2]) for any $0 \leqslant r \leqslant 1 \leqslant s$

(B.6) $\forall w \in H^s(]a,b[)$, $\|w - \pi_N w\|_{r,]a,b[} \leqslant C N^{r-s} \|w\|_{s,]a,b[}$,

and

(B.7) $\pi_N w(\pm 1) \square w(\pm 1)$.

Let us define the projection operators $\pi_{N(k),k}$, for any k in $\mathbb{N}$, $1 \leqslant k \leqslant K$, as being the projection operators from $H^1(\Lambda_k)$ onto $\mathbb{P}_{N(k)}(\Lambda_k)$. We deduce from (B.7) that the element $\tilde{\pi}_N^1 \Psi$ defined on each Λ_k by

$$\forall \, x \in \wedge_k \; , \; \tilde{\pi}_N^1 \, \Psi(x) = \pi_{N(k),k}(\Psi_{|\wedge_k})(x)$$

is an element of $\mathbb{P}_{N(k)}(\wedge_k) \cap H_0^1(\wedge)$ that satisfies, due to (B.6)

$$\| \Psi - \tilde{\pi}_N^1 \, \Psi \|_{r,\wedge} \leqslant C \, [\textstyle\sum_{k=1}^K N(k)^{2(r_k - s_k)} \, \| \Psi \|^2_{s_k,\wedge_k}]^{1/2} \; .$$

Then, (B.5) is proved.

Let us analyze now some properties of the operator of interpolation $\tilde{I}_{N,K}$ in $\tilde{\mathbb{P}}_{N,K}(\wedge)$ over the Gauss-Lobatto points. Since the degree $N(k)$ of the polynomials of approximation can vary with k, we must redefine the points of interpolation. It consists over each $\wedge_k$ of the $(\tilde{\xi}_{i,k})_{i=0,N(k)}$ defined in a similar way as in (2.16). Using the same techniques of decomposition of the interval $\overline{\wedge}$ in $\bigcup_{k=1}^K \overline{\wedge}_k$, we deduce from the classical results on the operator of interpolation in $\mathbb{P}_N(\wedge)$ over the Gauss-Lobatto points (ξ_i), $i=0,\dots,N$ (see [7; Thm. 3.2]) that

Theorem B.3 : *Let r be a K-tuple of real numbers, such that $r_k > 1/2$ .We have for any function Ψ in $\tilde{H}^r(\wedge)$*

(B.8) $\qquad \| \Psi - \tilde{I}_{N,K} \, \Psi \|_{0,\wedge} \leqslant C \, [\textstyle\sum_{k=1}^K N(k)^{1-2r_k} \, \| \Psi \|^2_{r_k,\wedge_k}]^{1/2} \; .$

B.2 The two-dimensional case.

In this paragraph, we shall combine the results of section B.1 with the classical results concerning the approximation theory related to the Fourier case. These results and the techniques we use are very close to those of [4; Appendix] and [17].

As in the previous section, we shall consider that the regularity of the function we want to approximate is different on the various Ω_k . To this hand, we associate with each Ω_k a couple $(N(k),M(k))$ of integers and consider the space of approximation

$$\mathscr{A}_h = \{ \, \varphi \in L^2(\Omega), \; \forall \, k, \; 1 \leqslant k \leqslant K \, , \; \varphi_{|\Omega_k} \in \mathbb{P}_{N(k)}(\wedge_k) \otimes S_{M(k)} \, \} \; .$$

Then, for any K-tuples r and s of positive real numbers, we consider the spaces

$$\tilde{H}^{r,s}(\Omega) = \{ \, \varphi \in L^2(\Omega), \; \forall \, k, \; 0 \leqslant k \leqslant K, \; \varphi_{|\Omega_k} \in H^{r_k}(\wedge_k; L^2(\Theta)) \cap L^2(\wedge_k; H^{s_k}_*(\Theta)) \, \} \; .$$

As in section 2 , we define also the spaces $\tilde{H}^{r,s}_*(\Omega)$ as being the closure of $C^\infty_*(\Omega)$ in $\tilde{H}^{r,s}(\Omega)$. We

shall use in the proofs some norms over Ω_k , the space $H^r(\wedge_k; L^2(\Theta))$ is provided with the norm $\| . \|_{r,0,\Omega_k}$; the space $L^2(\wedge_k; H^s_*(\Theta))$ is provided with the norm $\| \ \|_{0,s,\Omega_k}$.

First of all, let us consider the $L^2(\Omega)$-projection operator $\tilde{\Pi}_h$ onto $\mathscr{A}_h$ in $L^2(\Omega)$. We have

<u>Theorem B.4</u> : *Let* **r** *and* **s** *be two K-tuples of real numbers. For any function* Ψ *in* $\tilde{H}^{r,s}_*(\Omega)$

(B.9) $\quad \| \Psi - \tilde{\Pi}_h \Psi \|_{0,\wedge} \leqslant C [\sum_{k=1}^{K} (N(k)^{-2r_k} \| \Psi \|^2_{r_k,0,\Omega_k} + M(k)^{-2s_k} \| \Psi \|^2_{0,s_k,\wedge_k}]^{1/2}$.

$P\,roof$: Let us denote by $\Pi^{\#}_M$ the $L^2(\Theta)$-projection operator onto $S_M(\Theta)$. We recall the following inequality, valid for any $\alpha \geqslant 0$ (see [24])

(B.10) $\quad \forall \ v \in H^s_*(\Theta), \quad \| v - \Pi^{\#}_M v \|_{0,\Theta} \leqslant C \, M^{-s} \| v \|_{s,\Theta}$.

Then, as in the proof of theorem B.1, we derive from the definition of the $L^2(\Omega)$-projection operator that $\tilde{\Pi}_h$ coincides over Ω_k to the standard projection $\tilde{P}_k$ from $L^2(\Omega_k)$ onto $\mathbb{P}_{N(k)}(\wedge_k) \otimes S_{M(k)}$. It was proved in [4; theorem A.1] that

$$\| \Psi - \tilde{P}_k \Psi \|_{0,\Omega_k} \leqslant C [N(k)^{-2r_k} \| \Psi \|^2_{r_k,0,\Omega_k} + M(k)^{-2s_k} \| \Psi \|^2_{0,s_k,\wedge_k}]^{1/2} ,$$

we deduce that

$$\| \Psi - \tilde{\Pi}_h \Psi \|_{0,\Omega} \leqslant \sum_{k=1}^{K} \| \Psi - \tilde{P}_k \Psi \|^2_{0,\Omega_k}$$
$$\leqslant C [\sum_{k=1}^{K} (N(k)^{-2r_k} \| \Psi \|^2_{r_k,0,\Omega_k} + M(k)^{-2s_k} \| \Psi \|^2_{0,s_k,\wedge_k})]^{1/2} ,$$

and (B.9) is proved.

In the case where N and M are constant, we have simply for the operator Π_h of projection over $\mathbb{P}_N(\wedge) \otimes S_M$.

<u>Corollary B.1</u> : *Let* r *a nonnegative real number. We have for any function* Ψ *in* $H^r_*(\Omega)$

(B.11) $\quad \| \Psi - \Pi_h \Psi \|_{0,\Omega} \leqslant C (N^{-r} + M^{-r}) \| \Psi \|_{r,\Omega}$.

$P\,roof$: It suffices to notice that $H^r_*(\Omega)$ coincides with $H^r(\wedge; L^2(\Theta)) \cap L^2(\wedge; H^r_*(\Theta))$ (see [14; Chap. 4, Proposition 2.3]).

Next, we state the following inverse inequality. We first define, for any K-tuples **r** and **s** of

positive real numbers, the space

$$\mathscr{H}^{r,s}(\Omega) = \{\ \varphi \in L^2(\Omega),\ \varphi_{|\Omega_k} \in H^{r_k}(\Lambda_k; H^{s_k}_*(\Theta))\ $$

Lemma B.2 : *Let   r and* **s**, **r'** *and* **s'** *be K-tuples of real numbers, such that for any k,*

$1 \leqslant k \leqslant K, 0 \leqslant r_k \leqslant r'_k$ *and* $0 \leqslant s_k \leqslant s'_k$. *We have for any* Ψ *in* $\mathscr{A}_h$

$$(B.12) \quad \| \Psi \|_{\mathscr{H}^{r',s'}(\Omega)} \leqslant C \left[\sum_{k=1}^{K} N(k)^{4(r'_k - r_k)} M(k)^{2(s'_k - s_k)} \| \Psi \|^2_{H^{r_k}(\Lambda_k; H^{s_k}_*(\Theta))} \right]^{1/2} .$$

Proof : The following inverse inequality, valid for $s \leqslant s'$, is classical

$$(B.13) \quad \forall\ v \in S_M(\Theta)\ ,\quad \|v\|_{s',\Theta} \leqslant C\ M^{s'-s} \|v\|_{s,\Theta} .$$

We derive immediatly (B.12) from (B.3) and (B.13).

Corollary B.2 : *Let   r and r' be two nonnegative real numbers,* $r \leqslant r'$. *We have for any* Ψ *in*

$P_{N,K}(\Lambda) \otimes S_M(\Theta)$

$$(B.14) \quad \|v\|_{r',\Omega} \leqslant C\ (\ N^{2(r'-r)} + M^{r'-r}\ )\ [\ \sum_{k=1}^{K} \|v\|^2_{r,\Omega_k}\ ]^{1/2} .$$

Proof : Once more, we deduce (B.14) from lemma B.2 and from the fact that $H^r_*(\Omega)$ coincides with $H^r(\Lambda ; L^2(\Theta)) \cap L^2(\Lambda ; H^r_*(\Theta))$.

Now, for a given real number, we are interested in a-projection operator from $H^1_{0,*}(\Omega)$ onto $\mathscr{A}_{h,0} = \mathscr{A}_h \cap H^1_0(\Lambda ; L^2(\Theta))$.

Theorem B.5 : *There exists an operator* $\tilde{\pi}^1_h$ *from* $H^1_{0,*}(\Omega)$ *onto* $\mathscr{A}_{h,0}$ *verifying for any function* Ψ *in* $\tilde{H}^{r,r}_*(\Omega) \cap H^1_0(\Lambda ; L^2(\Theta))$ *where* **r** *is a K-tuple of real numbers* $\leqslant 1$

$$(B.15) \quad \| \Psi - \tilde{\pi}^1_h \Psi \|_{1,\Omega} \leqslant C [\ \sum_{k=1}^{K} [\ N(k)^{2(1-r_k)} + M(k)^{2(1-r_k)}\ ]\ \| \Psi \|^2_{r_k,\Omega_k}\ ]^{1/2} .$$

Proof : The great difficulty in this Theorem relay on the fact that the degree in the Fourier direction are different in each Ω_k but the resulting approximation has to be globally in $H^1(\Omega)$ which implies that at the boundary $a_k \times \Theta$, the trace of the approximation must be in $S_{\mu(k)}$ with $\mu(k) = \inf (M(k-1), M(k))$.

Our proof will be decomposed into 3 steps, and we first assume that $r_k \geqslant 2$.

1) It is a standard result of the trace theory (see e.g.[12]) to note that if Ψ belongs to $\tilde{H}^{r,r}_{\#}(\Omega) \cap H^1_0(\wedge; L^2(\Theta))$ the first trace can be defined over each $a_k \times \Theta$, more precisely, the mapping

$$\Psi \rightarrow \{ \psi_k = \Psi_{|a_k \times \Theta} \, , \; 1 \leqslant k \leqslant K+1 \}$$

is continuous from $\tilde{H}^{r,r}_{\#}(\Omega) \cap H^1_0(\wedge; L^2(\Theta)) \cap L^2(\wedge; H^1_{\#}(\Theta))$ into $\prod_{k=1}^{K+1} H_{\#}^{\sup(r_{k-1}, r_k)-1/2}(\Theta)$.

Besides, for any k, $2 \leqslant k \leqslant K$, there exists a continuous inverse mapping (see e.g. [12]) that associate to each ψ_k of $H_{\#}^{\sup(r_{k-1}, r_k)-1/2}(\Theta)$, an element $R_k(\psi_k)$ of $H_{\#}^{\sup(r_{k-1}, r_k)}(\wedge_{k-1} \times \Theta) \times H_{\#}^{\sup(r_{k-1}, r_k)}(\wedge_k \times \Theta)$, whose first trace over $a_k \times \Theta$ coincides with ψ_k and whose first trace over $a_{k-1} \times \Theta$ and $a_{k+1} \times \Theta$ is zero. In particular the continuity of each R_k that we can formulate as follows

(B.16) $\forall \, \psi_k \in H_{\#}^{\varrho_k}(\Theta)$, $\| R_k \psi_k \|_{\varrho_k + 1/2} \leqslant C \| \psi_k \|_{\varrho_k}$,

implies that the mapping R that associate to each Ψ of $\tilde{H}^{r,r}_{\#}(\Omega) \cap H^1_0(\wedge; L^2(\Theta))$ the element of $L^2(\Omega)$ defined as follows

$$\forall \, k, \, 2 \leqslant k \leqslant K-1, \; R\Psi_{|\Omega_k} = [R_k \Psi_{|a_k \times \Theta}]_{|\Omega_k} + [R_{k+1} \Psi_{|a_{k+1} \times \Theta}]_{|\Omega_k} \, ,$$

$$R\Psi_{|\Omega_1} = [R_2 \Psi_{|a_2 \times \Theta}]_{|\Omega_1} \, , \quad R\Psi_{|\Omega_K} = [R_K \Psi_{|a_K \times \Theta}]_{|\Omega_K} \, ,$$

satisfies $R\Psi$ belongs to $\tilde{H}^{s,s}_{\#}(\Omega) \cap H^1_0(\wedge; L^2(\Theta))$ with **s** defined by

(B.17) $\forall \, k, \, 1 \leqslant k \leqslant K, \, s_k = \inf(\sup(r_{k-1}, r_k), \sup(r_k, r_{k+1}))$,

where we have set $r_0 = r_{K+1} = 0$. Moreover, we also have

(B.18) $\forall \, k, \, 1 \leqslant k \leqslant K, \, (\Psi - R\Psi)_{|\Omega_k} \in H^1_0(\Omega_k)$.

2) It is an easy matter to find an element of $\mathcal{A}_{h,0}$ that approximate $\tilde{\Psi} = \Psi - R\Psi$. Indeed it suffices to take over each $\wedge_k$, $(\pi^1_{N(k)} \otimes \pi^{\#}_{M(k)})\tilde{\Psi}$ since this element vanishes over each elemental boundary. Moreover, it verifies

(B.19) $\| \tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^{\#}_{M(k)})\tilde{\Psi} \|_{1, \Omega_k} \leqslant C \, (\; \| \tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^{\#}_{M(k)})\tilde{\Psi} \|_{L^2(\wedge_k; H^1_{\#}(\Theta))}$

$$+ \| \tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^{\#}_{M(k)})\tilde{\Psi} \|_{H^1(\wedge_k; L^2(\Theta))} \;) \; .$$

We write the first term

$$\|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{L^2(\wedge_k; H^1_\#(\Theta))} \leq \|(Id \otimes \pi^\#_M)(\tilde{\Psi} - (\pi^1_N \otimes Id)\tilde{\Psi})\|_{L^2(\wedge; H^1_\#(\Theta))}$$
$$+ \|\tilde{\Psi} - (Id \otimes \pi^\#_M)\tilde{\Psi}\|_{L^2(\wedge; H^1_\#(\Theta))} \cdot$$

Now, we recall that $\pi^\#_M$ commutes with the differentiation operators and satisfies for $0 \leq r \leq s$ [24]:

(B.20) $\quad \forall v \in H^s_\#(\Theta), \quad \|v - \pi^\#_M v\|_{r,\Theta} \leq C M^{r-s} \|v\|_{s,\Theta} \cdot$

Using (B.5) for $r = 0$, (B.20) for $r = s = 1$ and (B.20) for $r = 1$ gives for $s \geq 2$,

$$\|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{L^2(\wedge_k; H^1_\#(\Theta))} \leq C \left(N(k)^{1-r_k} \|\tilde{\Psi}\|_{H^{r_k-1}(\wedge_k; H^1_\#(\Theta))} \right.$$
$$\left. + M(k)^{1-r_k} \|\tilde{\Psi}\|_{L^2(\wedge_k; H^{r_k}_\#(\Theta))} \right) \cdot$$

Moreover, it is standard to note that, for r_k

(B.21) $\quad H^{r_k}_\#(\Omega_k) \subset H^{r_k-1}(\wedge_k; H^1_\#(\Theta)).$

Hence, we obtain for $s \geq 2$

(B.22) $\quad \|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{L^2(\wedge_k; H^1_\#(\Theta))} \leq C (N(k)^{1-r_k} + M(k)^{1-r_k}) \|\tilde{\Psi}\|_{r_k, \Omega_k} \cdot$

In the same way, we write the second term

$$\|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{H^1(\wedge_k; L^2(\Theta))} \leq \|(Id \otimes \pi^\#_{M(k)})(\tilde{\Psi} - (\pi^1_{N(k)} \otimes Id)\tilde{\Psi})\|_{H^1(\wedge_k; L^2(\Theta))}$$
$$+ \|\tilde{\Psi} - (Id \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{H^1(\wedge_k; L^2(\Theta))} \cdot$$

Using (B.5) with $r = 1$, (B.20) with $r = s = 0$ and (B.20) with $r = 1$ gives

$$\|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{H^1(\wedge_k; L^2(\Theta))} \leq C \left(N(k)^{1-r_k} \|\tilde{\Psi}\|_{H^{r_k}(\wedge_k; L^2(\Theta))} \right.$$
$$\left. + M(k)^{1-r_k} \|\tilde{\Psi}\|_{H^1(\wedge_k; H^{r_k-1}_\#(\Theta))} \right) \cdot$$

so that, by (B.21),

(B.23) $\quad \|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{H^1(\wedge_k; L^2(\Theta))} \leq C (N(k)^{1-r_k} + M(k)^{1-r_k}) \|\tilde{\Psi}\|_{r_k, \Omega_k} \cdot$

Finally, the inequalities (B.19)(B.22)(B.23) imply

(B.24) $\quad \|\tilde{\Psi} - (\pi^1_{N(k)} \otimes \pi^\#_{M(k)})\tilde{\Psi}\|_{1, \Omega_k} \leq C (N(k)^{1-r_k} + M(k)^{1-r_k}) \|\tilde{\Psi}\|_{r_k, \Omega_k} \cdot$

(cf. [14; Chap. I, Théorème 5.1]).

3) The same technique can be applied to the restriction of each $R_K\Psi|_{a_K\times\Theta}$ to Ω_{k-1} and to Ω_k. But here we know that the element $R_K\Psi|_{a_K\times\Theta}$ has the best regularity of Ψ over Ω_{k-1} or Ω_k. We approximate $R_K\Psi|_{a_K\times\Theta}$ by an element of $\mathbb{P}_{n(k)}(\wedge_{k-1})\times S_{m(k)}$ over Ω_{k-1} and by an element of $\mathbb{P}_{n(k)}(\wedge_k)\times S_{m(k)}$ over Ω_k where we have set $n(k) = \inf(N(k-1),N(k))$ and $m(k) = \inf(M(k-1),M(k))$. Summing up the resulting approximations we derive an approximation $\pi_h(\Psi)$ of $R\Psi$ in $\mathcal{A}_{h,0}$ that satisfies

$$(B.25)\quad \|R\Psi - \pi_h(\Psi)\|_{1,\Omega_k} \leqslant C\,(n(k)^{1-\sup(r_{k-1},r_k)} + m(k)^{1-\sup(r_{k-1},r_k)})\,\|R_K\Psi|_{a_K\times\Theta}\|_{\sup(r_{k-1},r_k)}$$

$$+\,n(k)^{1-\sup(r_k,r_{k+1})} + m(k+1)^{1-\sup(r_k,r_{k+1})})\,\|R_{K+1}\Psi|_{a_{K+1}\times\Theta}\|_{\sup(r_k,r_{k+1})}$$

Finally, we deduce that the polynomial $Q_h = (\pi^1_{N(k)}\otimes\pi^\#_{M(k)})\tilde\Psi + \pi_h(\Psi)$ satisfies

$$\|\Psi - Q_h\|_{1,\Omega} = \|\tilde\Psi + R\Psi - Q_h\|_{1,\Omega} \leqslant \|\Phi\Psi - (\pi^1_{N(k)}\otimes\pi^\#_{M(k)})\tilde\Psi\|_{1,\Omega} + \|R\Psi - \pi_h(\Psi)\|_{1,\Omega}$$

and the result (B.15) follows, when all the p_k are $\geqslant 2$, from (B.17)(B.18)(B.24) and (B.25) since

$$\|\Psi - \tilde\pi_h\Psi\|_{1,\Omega} \leqslant \|\Psi - Q_h\|_{1,\Omega} \leqslant C\left[\sum_{k=1}^K [N(k)^{2(1-r_k)} + M(k)^{2(1-r_k)}]\,\|\Psi\|^2_{r_k,\Omega_k}\right]^{1/2}\ .$$

The projection operator $\tilde\pi_h$ is stable in the $H^1(\Omega)$ norm then we have

$$\|\Psi - \tilde\pi_h\Psi\|_{1,\Omega} \leqslant \|\Psi\|_{1,\Omega}\ .$$

The general result follows by using the main theorem of interpolation between Hilbert spaces of [14].

We wish to obtain an error estimate for the interpolation operator. Let us denote by $x_{\tilde i_k}$, $\tilde i_k = (i_k,j_k)$, $0\leqslant i_k\leqslant N(k)$, $0\leqslant j_k\leqslant 2M(k)$, the points $(\tilde\xi_{i,k},\tilde\theta_{j,k})$ where $\tilde\theta_{j,k} = -\pi + 2j_k\pi/(2M(k)+1)$. Let us defined the operator I_h from $C^\circ(\overline\Omega)$ into $\mathcal{A}_h$ by

$$(B.26)\quad \forall v\in C^\circ(\overline\Omega),\ \forall k,\ 1\leqslant k\leqslant K,\ \forall \tilde i_k = (i_k,j_k),\quad I_h v(x_{\tilde i_k}) = v(x_{\tilde i_k})\ .$$

<u>Theorem B.6</u> : *Let* **r** *and* **s**, **r'** *and* **s'** *be* K-*tuples of real numbers greater than* 1/2. *We have for any function* v *in* $H^{r,s}(\Omega) \cap \mathcal{H}^{r,s}(\Omega)$

$$(B.27) \quad \|v - I_h v\|_{0,\Omega} \leqslant C \left[\sum_{k=1}^{K} (N(k))^{1-2r_k} \|v\|^2_{H^{r_k}(\wedge_k; L^2(\Theta))} + M(k)^{1-2s_k} \|v\|_{L^2(\wedge_k; H^{s_k}(\Theta))} \right.$$
$$\left. + N(k)^{1-2r'_k} M(k)^{1-2s'_k} \|v\|_{H^{r'_k}(\wedge_k; H^{s'_k}(\Theta))} \right)$$

Proof : Let us denote by $I_M^{\#}$ the operator from $C^\circ(\overline{\Theta})$ into $S_M(\Theta)$ defined by

$$(B.28) \quad \forall v \in C^\circ(\overline{\Theta}), \forall j, 0 \leqslant j \leqslant 2M, \quad I_M^{\#} v(\theta_j) = v(\theta_j) .$$

Since over each Ω_k, I_h is equal to $I_{N(k)} \otimes I_{M(k)}^{\#}$, we have

$$\|v - I_h v\|_{0,\Omega_k} \leqslant (\|v - (\text{Id} \otimes I_{M(k)}^{\#})v\|_{L^2(\wedge_k; L^2(\Theta))} + \|v - (I_{N(k)} \otimes \text{Id})v\|_{L^2(\wedge_k; L^2(\Theta))}$$
$$+ \| (\text{Id} - (\text{Id} \otimes I_{M(k)}^{\#}))(v - (I_{N(k)} \otimes \text{Id})v)\|_{L^2(\wedge_k; L^2(\Theta))} .$$

Then, we deduce the theorem from (B.8) and the classical result (see [7; Theorems 1.2]), valid for s > 1/2,

$$(B.29) \quad \forall v \in H_*^s(\Theta), \quad \|v - (\text{Id} \otimes I_M^{\#})v\|_{0,\Theta} \leqslant C M^{-s} \|v\|_{s,\Theta} .$$

In the more simple case where $N(k)$ and $M(k)$ are independent of k, we have

<u>Corollary B.3</u> : *Let* r *be a real number* > 1. *We have for any function* v *in* $H_*^r(\Omega)$ *and for any* μ > 1/2

$$(B.30) \quad \|v - I_h v\|_{0,\Omega} \leqslant C (M^{-r} + (1 + M^{-\mu}N^{\mu})N^{1/2-r}) \|v\|_{r,\Omega} .$$

Proof : Using Theorem B.5 with $r_k = s_k = r'_k + s'_k$, independent of k together with (B.21), yields

$$\|v - I_h v\|_{0,\Omega} \leqslant C (M^{-r} + N^{1/2-r} + M^{-r'} N^{1/2-r+r'}) \|v\|_{r,\Omega} .$$

This inequality is valid for any $r' = \mu > 1/2$.

The end of this section being more related to our analysis than the previous general results, we shall suppose that $N(k)$ and $M(k)$ are independent of k ; besides, it is clear, though tedious to write and read , how the general result would exist.

We want to estimate the difference.

(B.31) $E(f,w) = (f,w) - (f,w)_{h,GL}$

for any f in $C^\circ(\overline{\Omega})$ and w in $P_{N,K}(\wedge) \otimes S_M$. Let us denote by h* the couple (N-1,M), we can prove

Lemma B.3 : *For any f in $C^\circ(\overline{\Omega})$ and any w in $P_{N,K}(\wedge) \otimes S_M$ we have*

(B.32) $|E(f,w)| \leqslant C [\| f - \Pi_{h*} f \|_{0,\Omega} - \| f - I_h f \|_{0,\Omega}] \| w \|_{0,\Omega}$.

Proof : From $(2.15)_{GL}$ we deduce that

$E(f,w) = E(f - \Pi_{h*} f , w)$,

hence, from (3.28)

$|E(f,w)| \leqslant C [\| f - \Pi_{h*} f \|_{0,\Omega} + \| I_h f - \Pi_{h*} f \|_{0,\Omega}] \| w \|_{0,\Omega}$

$\leqslant C [\| f - \Pi_{h*} f \|_{0,\Omega} + \| f - I_h f \|_{0,\Omega}] \| w \|_{0,\Omega}$.

Using (B.9) and (B.27) yields immediatly

Theorem B.7 : *Let r and s, r' and s' be real numbers greater than 1/2. For any w in*
$P_{N,K}(\wedge) \otimes S_M$ *and any f in* $H^r(\wedge; L^2(\Theta)) \cap L^2(\wedge; H^s(\Theta)) \cap H^{r'}(\wedge; H^{s'}(\Theta))$

(B.33) $|E(f,w)| \leqslant C \| w \|_\Omega (N^{1/2-r} \| v \|_{H^r(\wedge; L^2(\Theta))} + M^{-s} \| v \|_{L^2(\wedge; H^s(\Theta))}$

$+ N^{1/2-r'} M^{-s'} \| v \|_{H^{r'}(\wedge; H^{s'}(\Theta))})$.

Finally (B.9) and (B.30) gives

Corollary B.4 : *Let r be a real number > 1. For any w in $P_{N,K}(\wedge) \otimes S_M$ and any f in*
$H_{\#}^r(\Omega)$ *and for any $\mu > 1/2$*

(B.34) $|E(f,w)| \leqslant C \| w \| (M^{-r} + (1+M^{-\mu}N^\mu)N^{1/2-r}) \| f \|_{r,\Omega}$.

Figure caption

Figure 1. The error in the velocity $u_N = (u_N, v_N)$ and the pressure p_N as a function of the total number of degrees of freedom (Gauss Legendre Lobatto points) in the x-direction when solving the discrete equation (4.12) corresponding to the test problem (4.13). The total interval $\wedge$ =]-1,1[is divided into two equal spectral elements of lengh $\wedge_1$ =]-1,0[and $\wedge_1$ =]0,1[, i.e. K = 2. Exponential convergence is obtained (the plot in log-lin).

Figure 2. The error in the pressure p_N as a function of the total number of degrees of freedom (Gauss Legendre Lobatto points) in the x-direction when solving the discrete equation (4.12) corresponding to the test problem (4.14). The error is given for γ = 3 and for γ = 5. The total interval $\wedge$ =]-1,1[is not divided into any subintervals, i.e. K = 1. Algebraic convergence is achieved asymptotically (the plot is log-log), although initially for small N faster convergence is achieved.
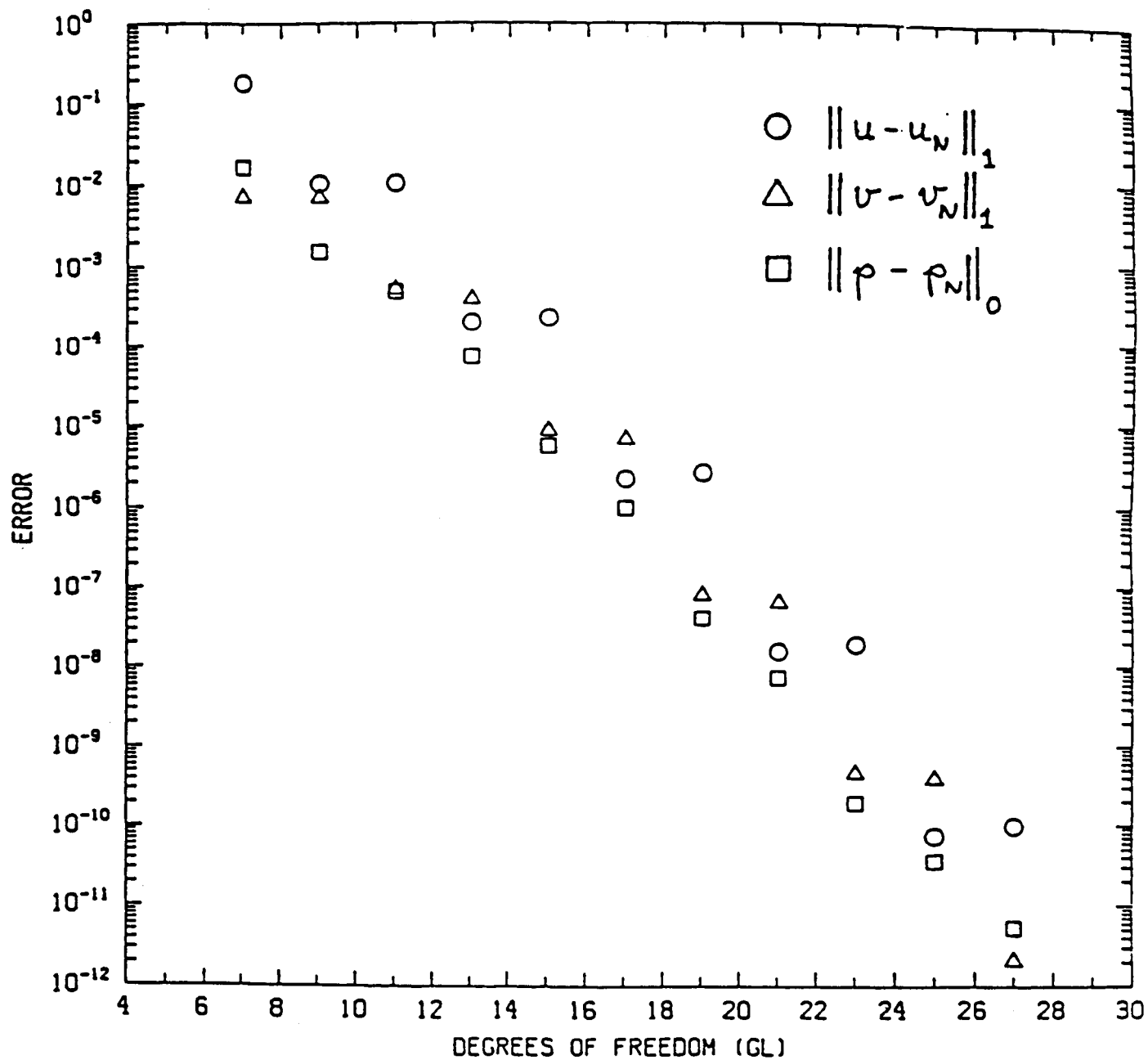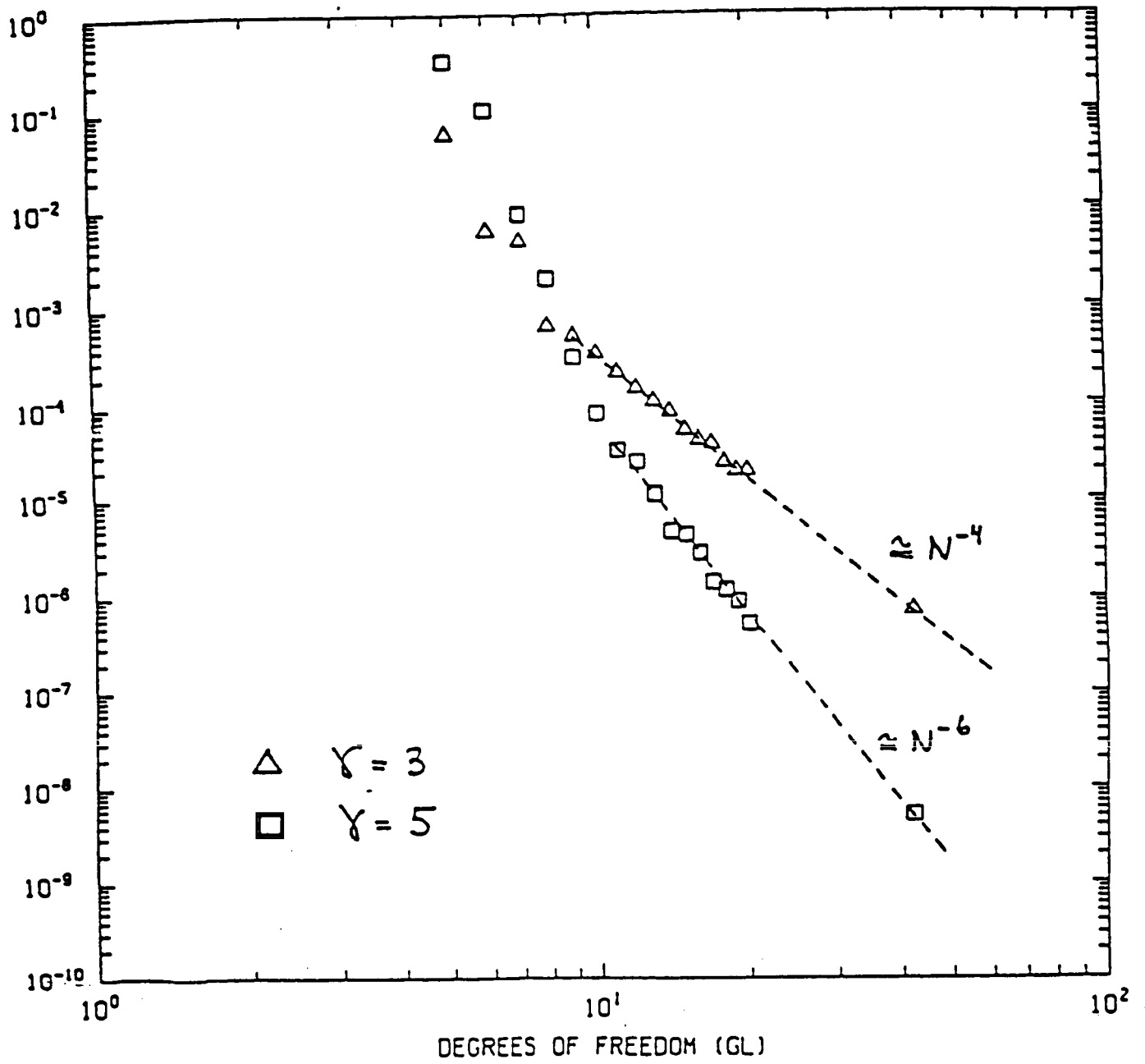
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Final Report

16. Abstract

In this paper we propose a new method for the spectral element simulation of incompressible flow. This method consists in a well-posed optimal approximation of the steady Stokes problem with no spurious modes in the pressure. The resulting method is analyzed, and numerical results are presented for a model problem.

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